Image Reconstruction: Back Projection

Back Projection

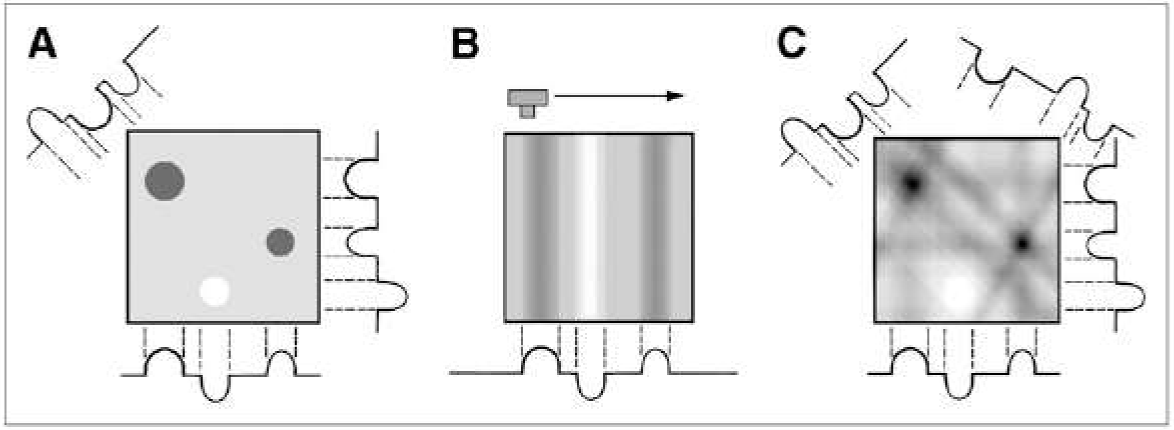


FIGURE 5. (A) Backprojection reconstruction for simple phantom containing 3 objects with different attenuation values. (B) For each view, attenuation values are simply divided evenly along their ray paths. Summing backprojected views from several angles builds image. (C) Four views of phantom are summed. Although this method is efficient, images reconstructed with backprojection exhibit considerable blurriness.

Back Projection

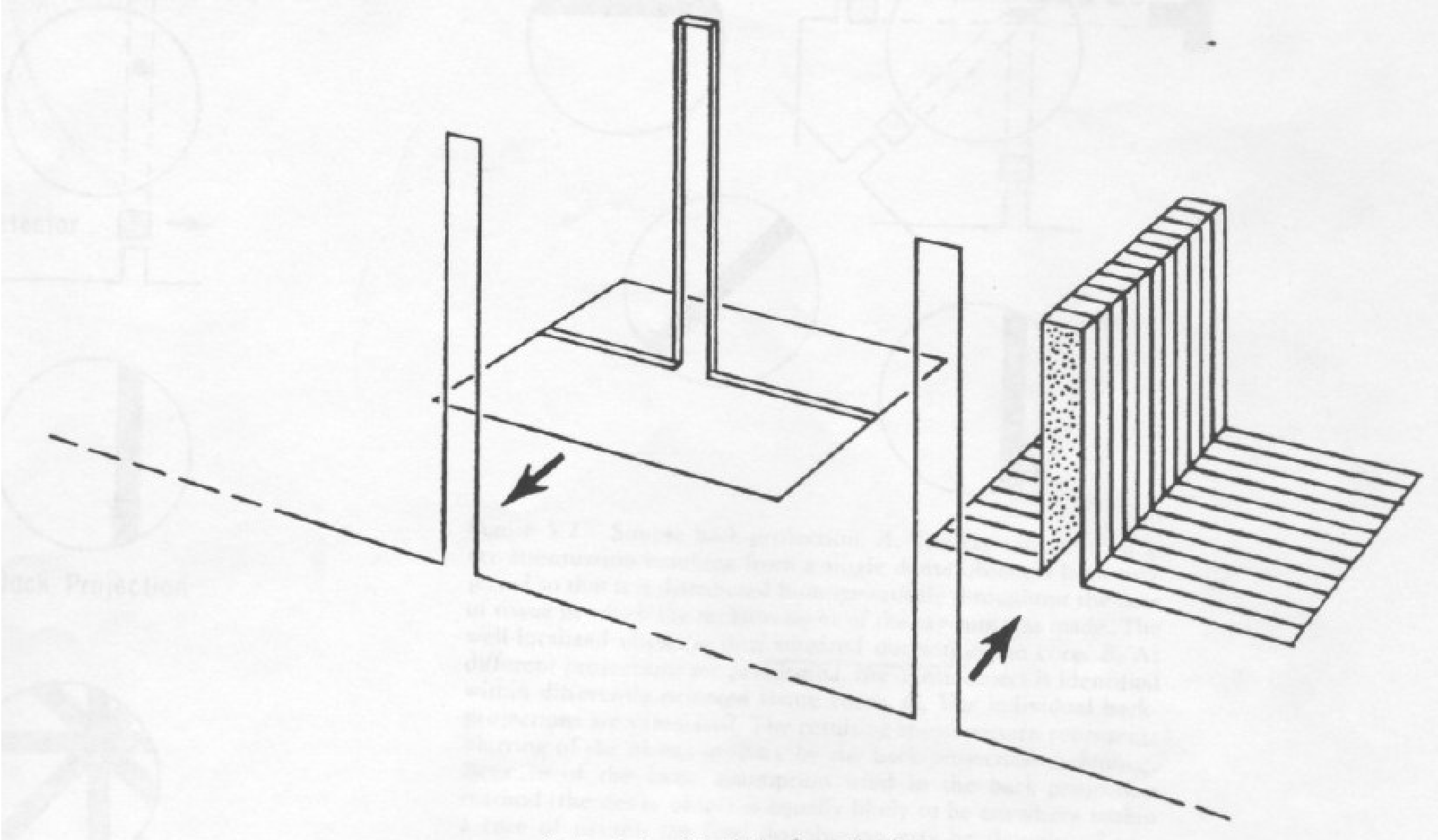


Figure 5.1. Simple back-projection.

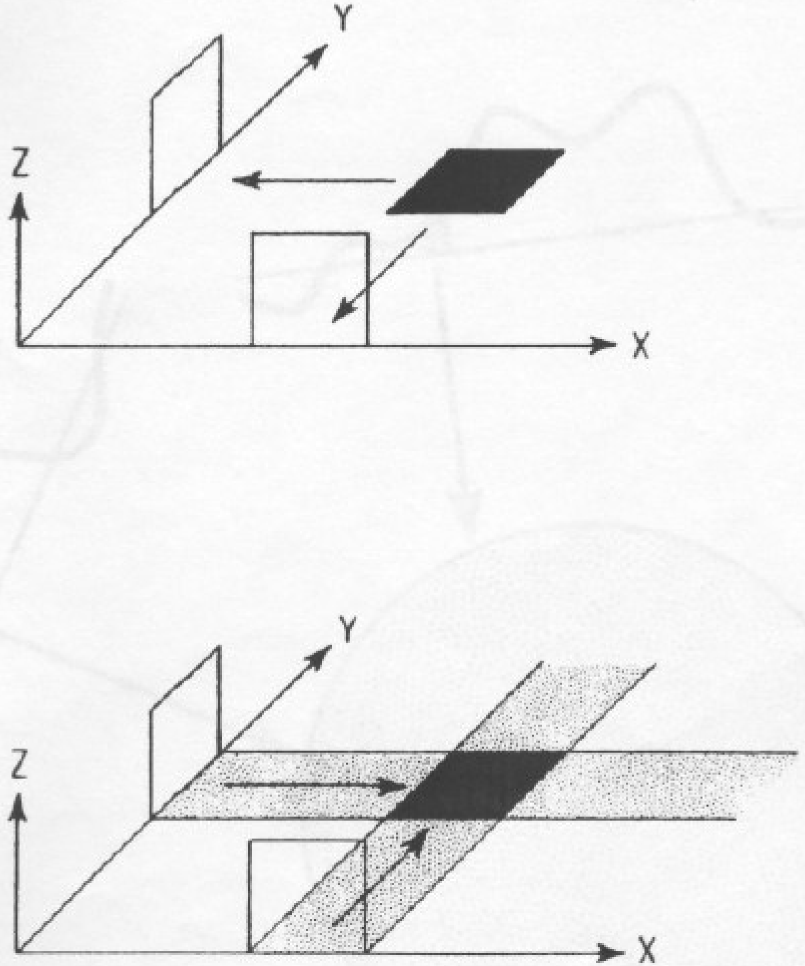


Figure 5.3. Simple back-projection. A dense square in the XY plane is projected onto the X and the Y axes (two independent projections). When the vertical projections are back-projected onto the XY plane, a cross-pattern is obtained. The intersection of the two back-projections corresponds to the original high density square.

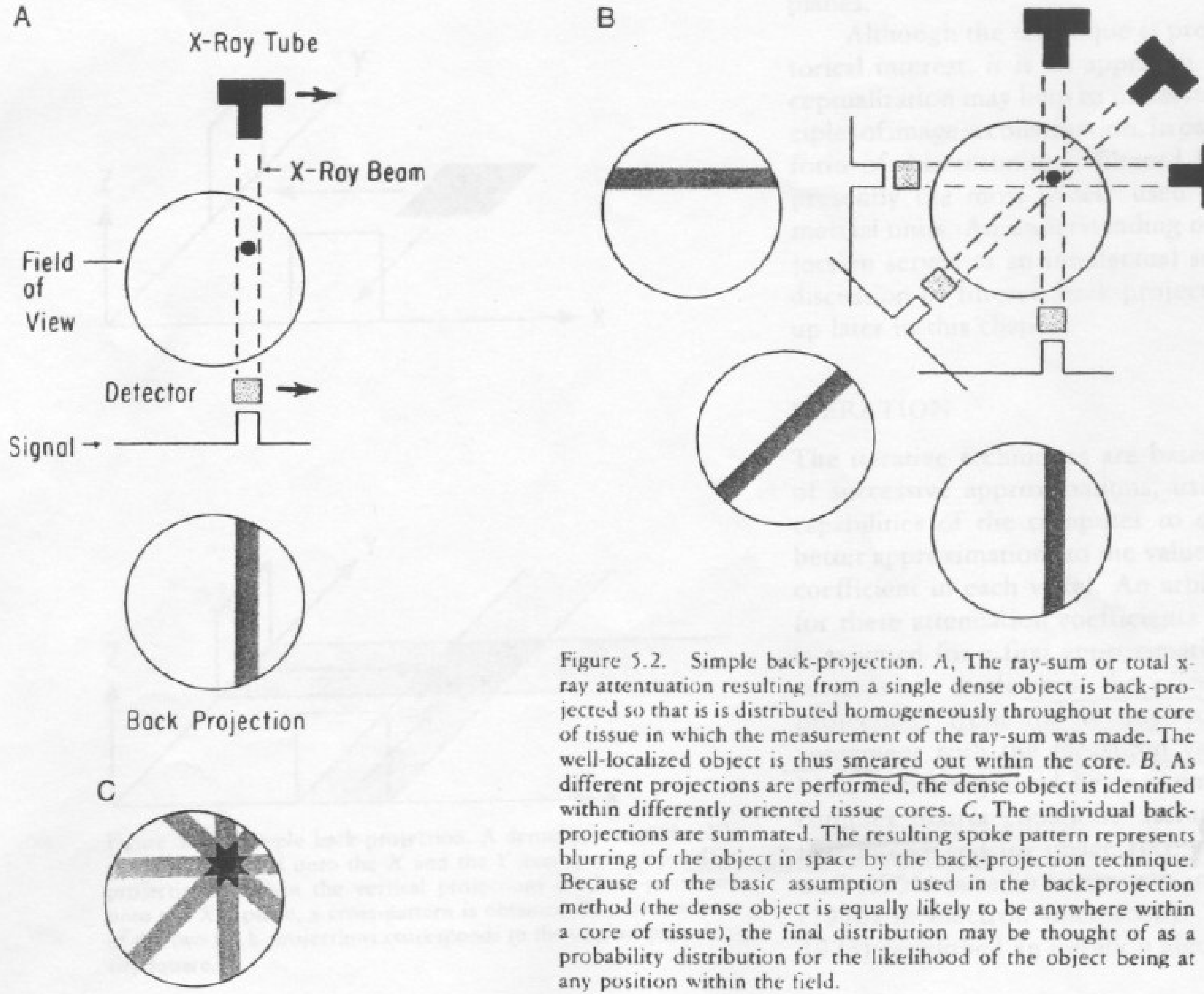


Figure 5.2. Simple back-projection. *A*, The ray-sum or total x-ray attentuation resulting from a single dense object is back-projected so that is is distributed homogeneously throughout the core of tissue in which the measurement of the ray-sum was made. The well-localized object is thus smeared out within the core. *B*, As different projections are performed, the dense object is identified within differently oriented tissue cores. *C*, The individual back-projections are summated. The resulting spoke pattern represents blurring of the object in space by the back-projection technique. Because of the basic assumption used in the back-projection method (the dense object is equally likely to be anywhere within a core of tissue), the final distribution may be thought of as a probability distribution for the likelihood of the object being at any position within the field.

Computed Tomography: Reconstruction

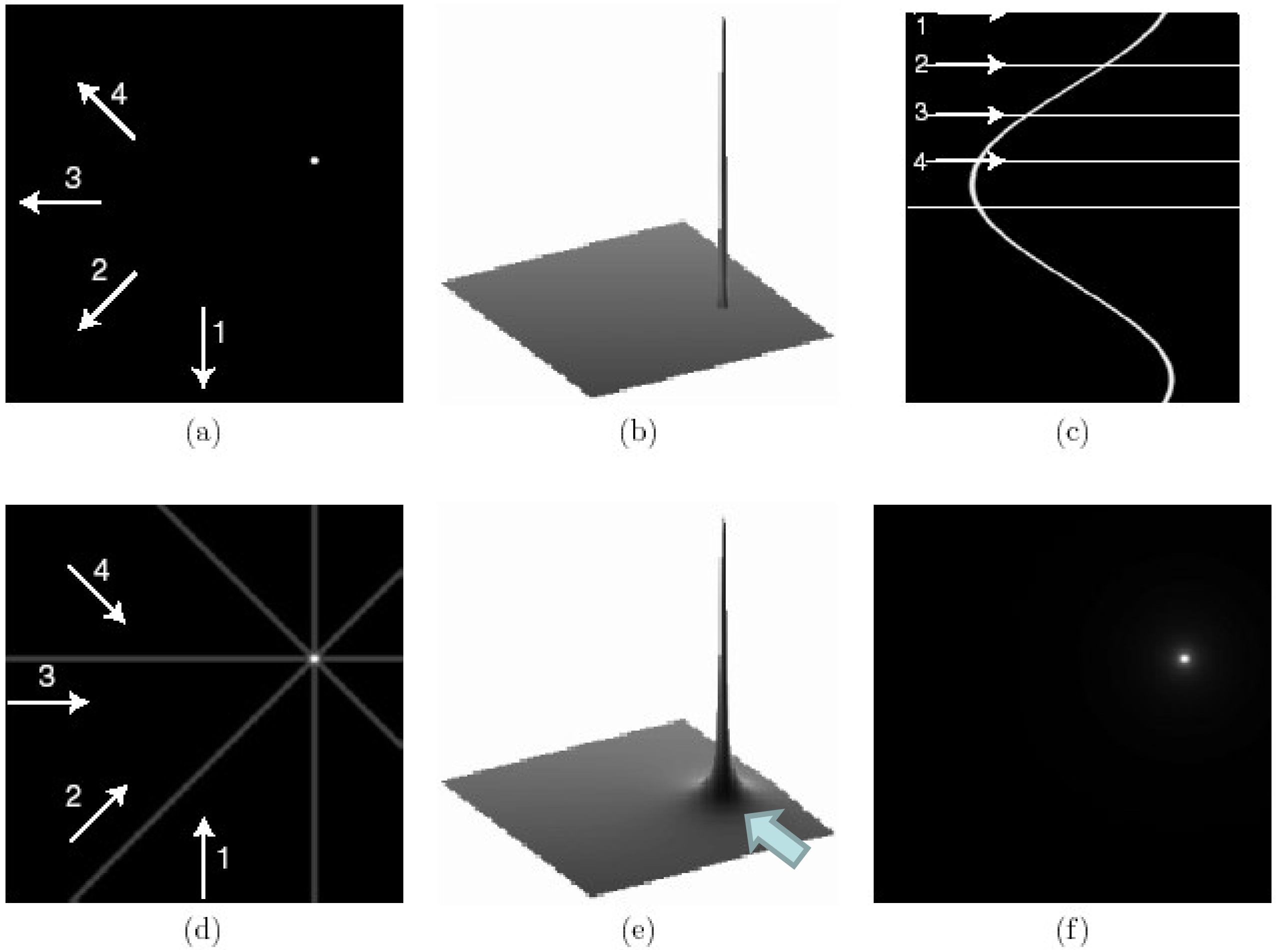


Figure 5.7: *(a-b) Image and surface plot of a distribution $\mu(x, y)$ containing one single dot. The arrows indicate four arbitrary projection directions. (c) 360°-sinogram obtained by projecting $\mu(x, y)$. The arrows indicate the views that correspond to the four projection directions in (a). (d) Back-projection (see section ??) of the four views chosen in (a). (e-f) Surface plot and image of the straightforward back-projection of the entire sinogram in (c).*

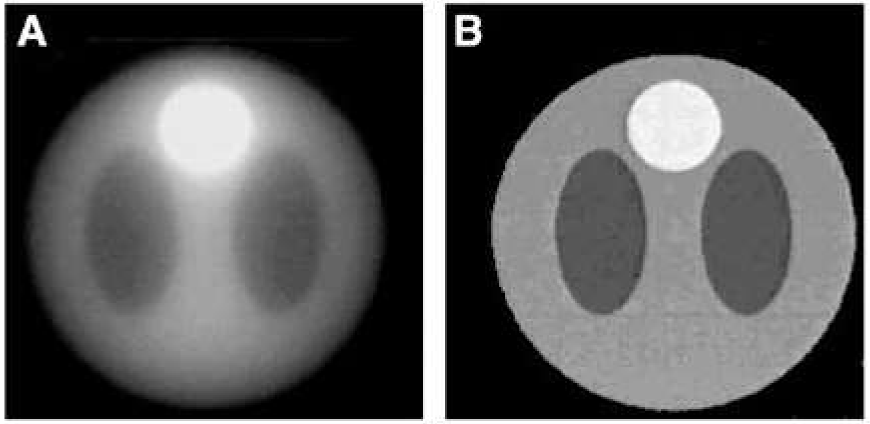


FIGURE 6. FBP. Mathematic phantom image reconstructed without (A) and with (B) filtering. FBP effectively reconstructs high-quality images. Adapted from S. Napel.

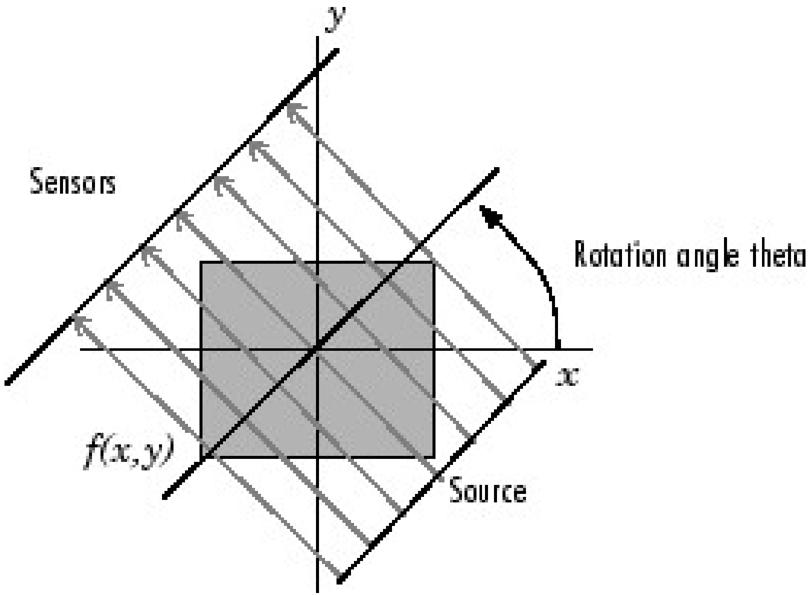
Image Reconstruction: Filtered Back Projection

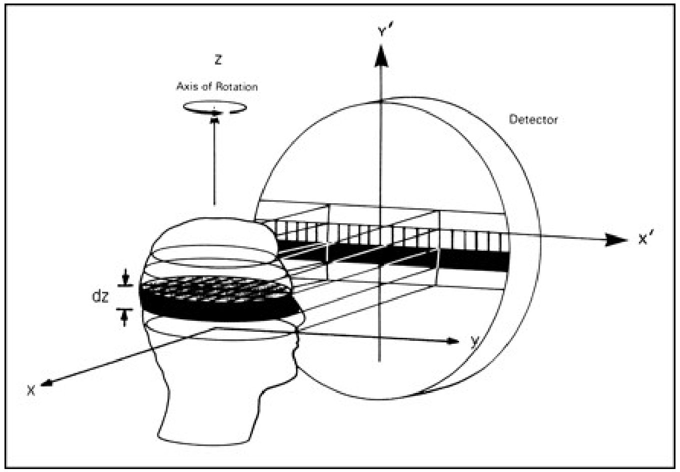
Radon Transformation

Radon Transformation

- Radon transform in 2-D.
- Named after the Austrian mathematician Johann Radon
- RT is the integral transform consisting of the integral of a function over straight lines.
- The inverse of RT is used to reconstruct images from medical computed tomography scans.

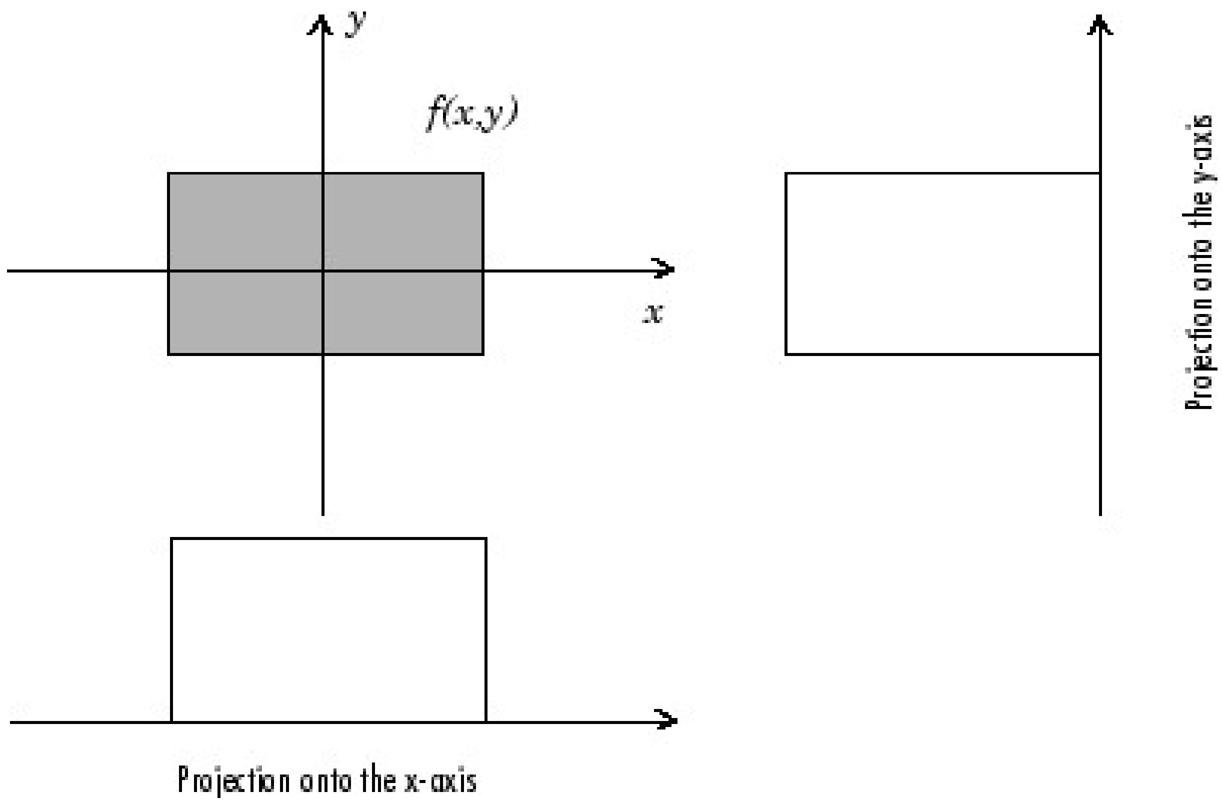
Paralle-beam Projection





- A projection of a 2-D image f(x,y) is a set of line integrals.
- To represent an image, RT takes multiple, parallel-beam projections of the image from different angles by rotating the source around the center of the image.

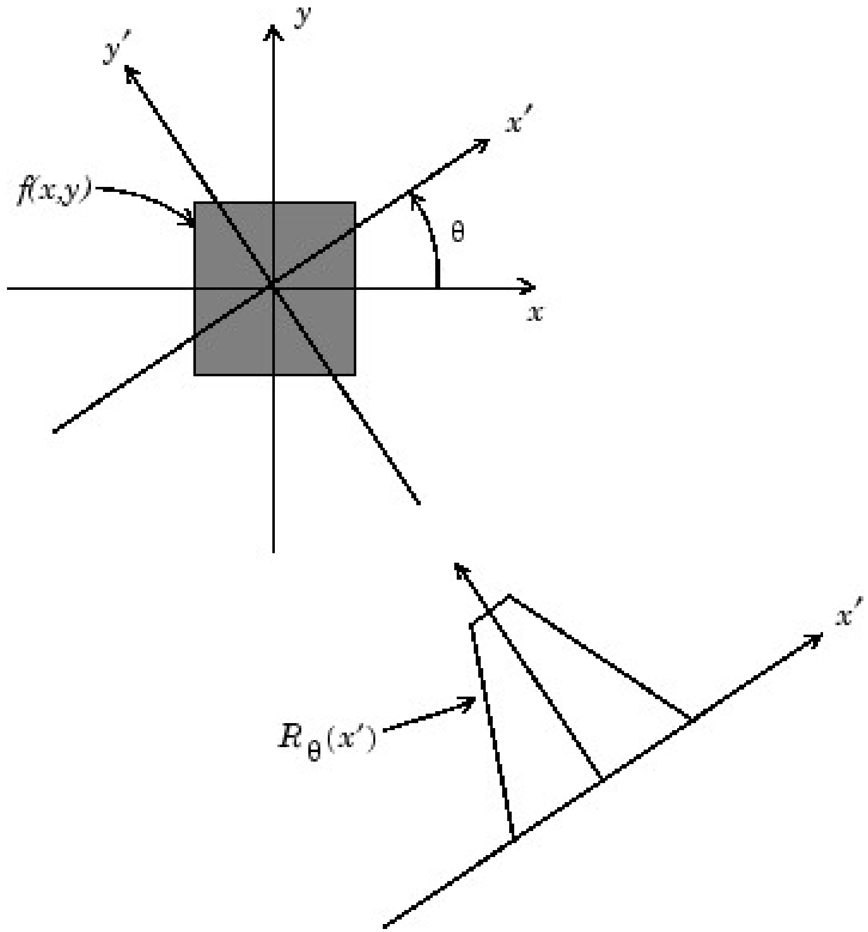
- For instance, the line integral of f(x,y) in the vertical direction is the projection of f(x,y) onto the x-axis.



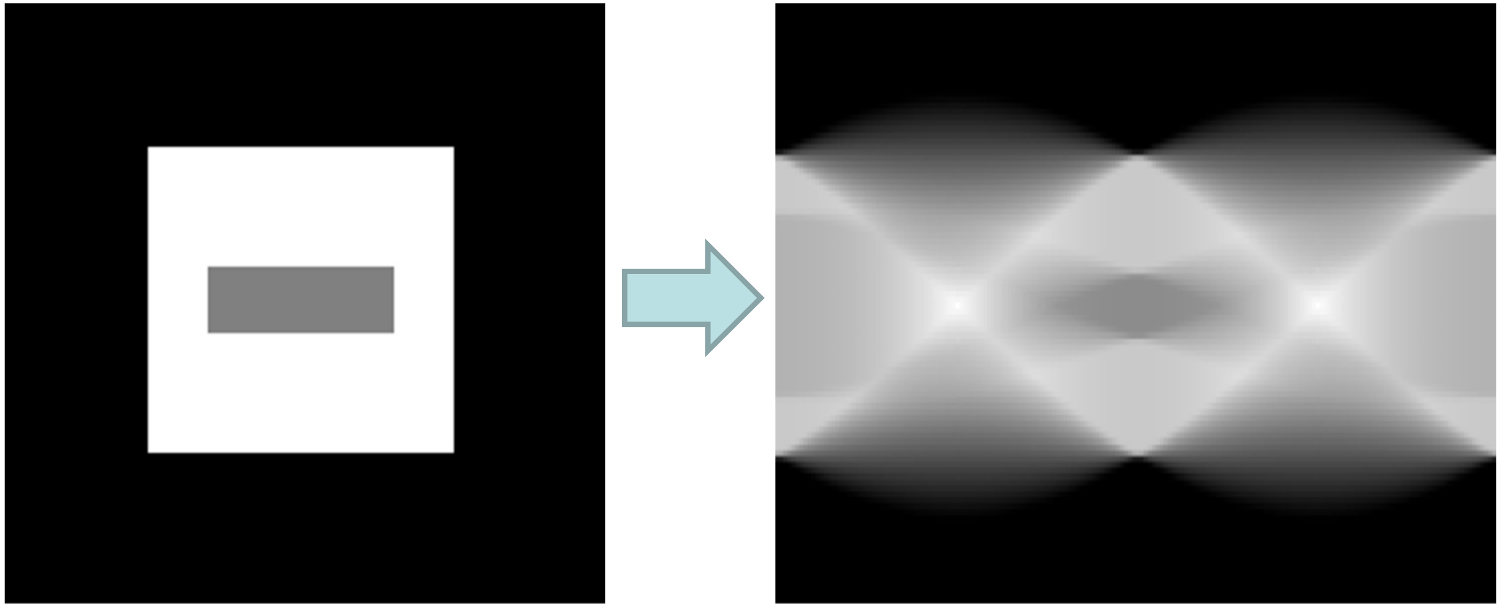
Math. and Geometry of the Radon Transform

$$R_\theta(x') = \int_{-\infty}^{\infty} f(x'\cos\theta - y'\sin\theta,\ x'\sin\theta + y'\cos\theta)\,dy'$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



RT Example

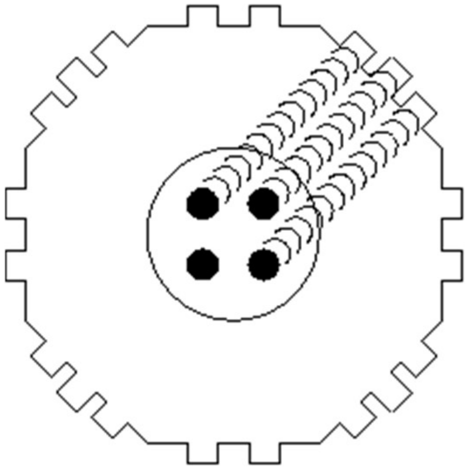


Filtered Back Projection

Backprojection

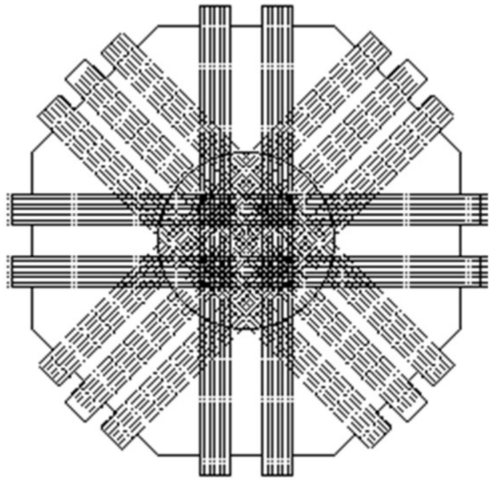
- A problem with backprojection is the blurring (star-like artifacts) that happens in the reconstructed image.

A



Projection

B



Backprojection

Filtered Backprojection

- To remove the blurring, an optimal way is to apply a high-pass filter to eliminate these artifacts.
- Thus combine backprojection with high-pass filtering = filtered backproejction.

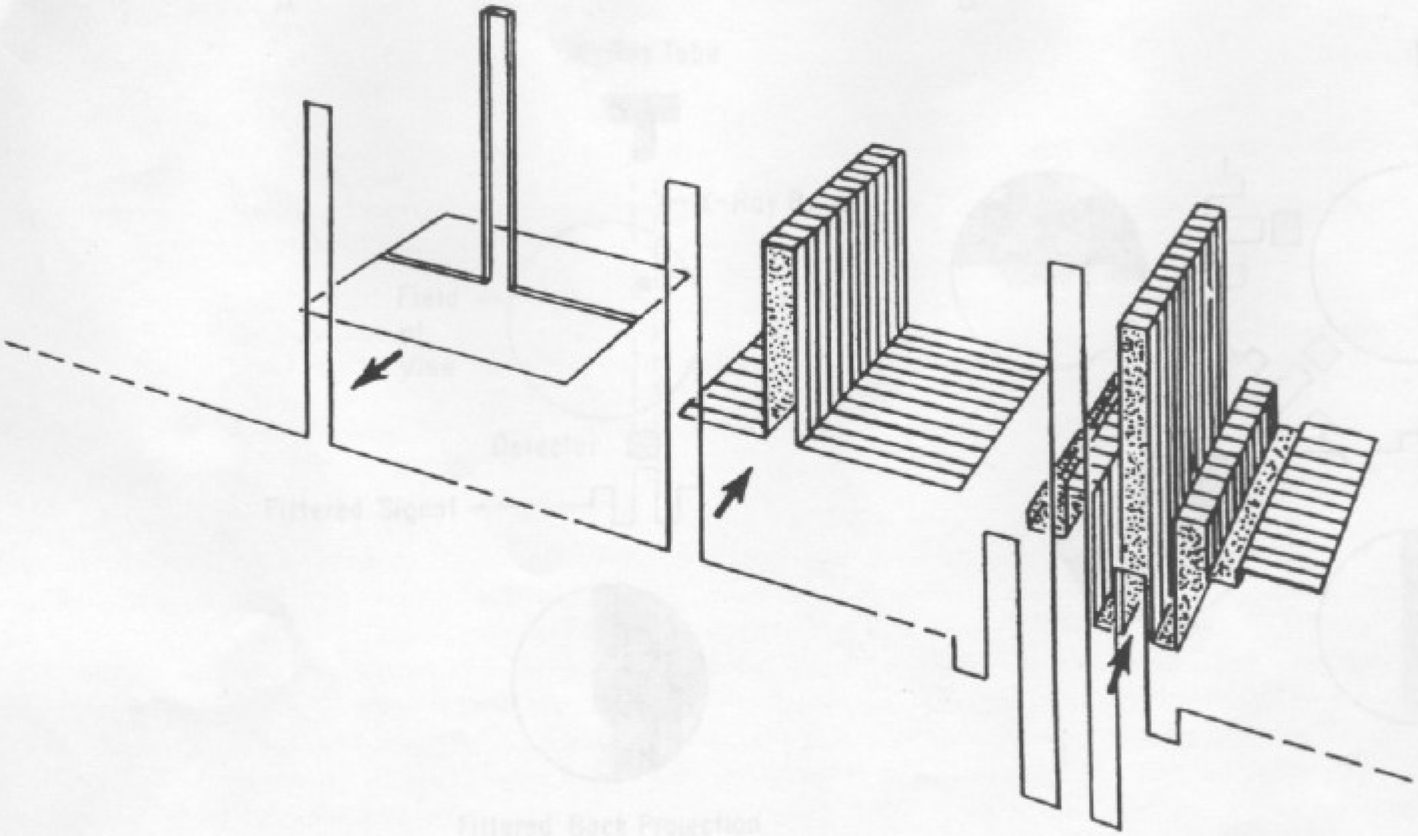


Figure 5.11. The difference between simple and filtered back-projection is illustrated. In simple back-projection, the ray-sum is merely projected back directly onto the ray, distributing its value equally among all the elements in the ray (middle illustration). In filtered back-projection, the ray-sum is filtered by a function that adds negative and positive components (far right).

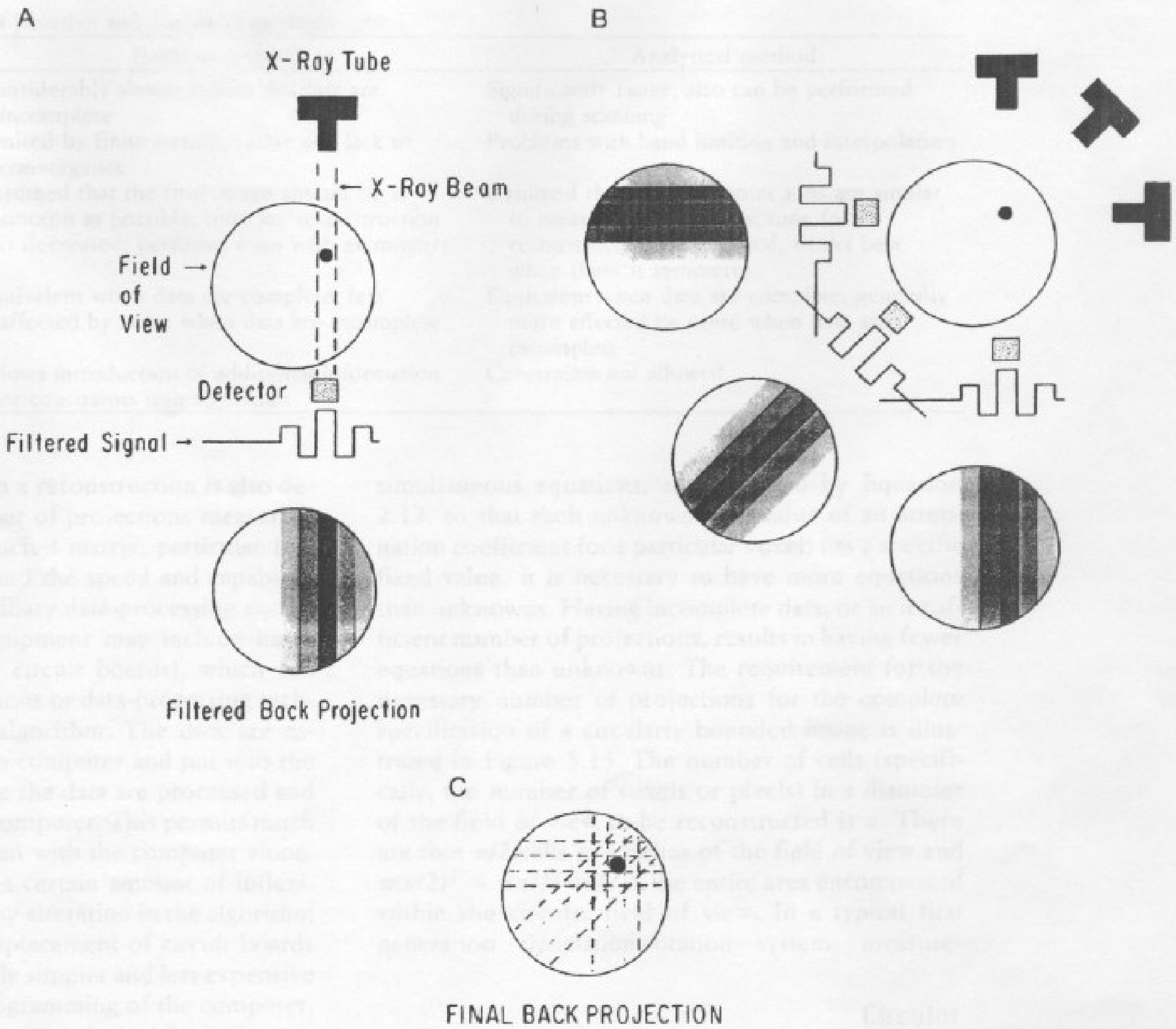
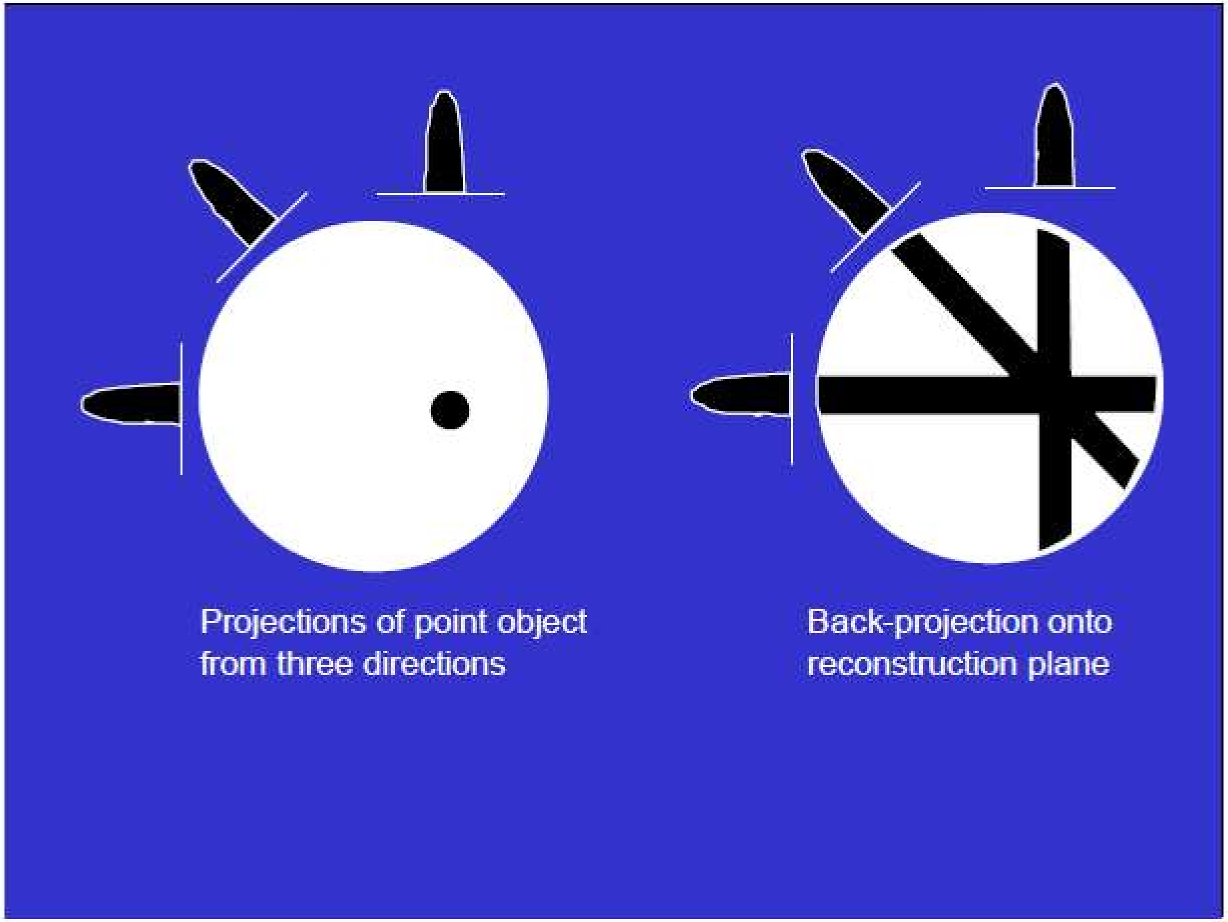
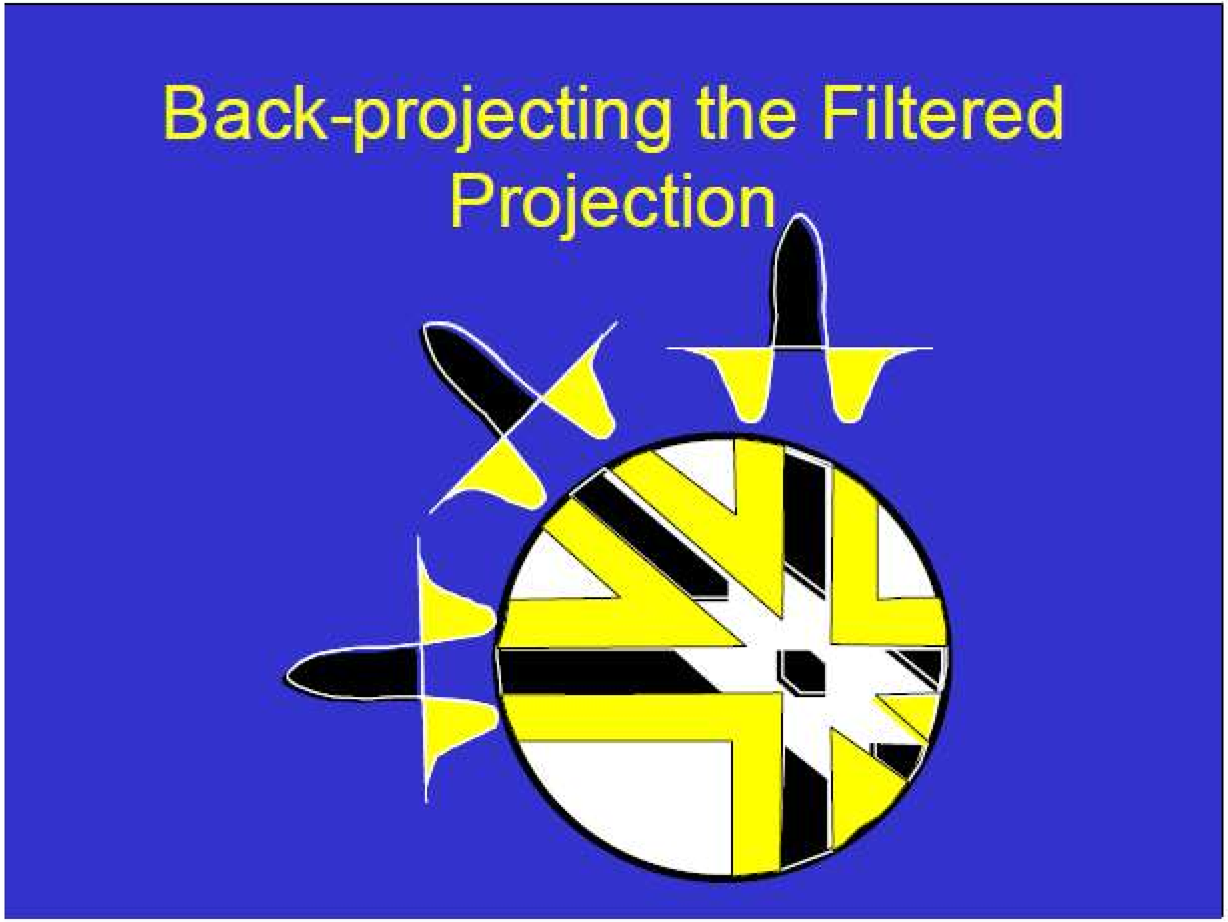


Figure 5.12. Filtered back-projection. *A*, The ray-sum or total x-ray attenuation resulting from a single dense object is first filtered by a function with negative and positive components. The filtered ray-sum is then back-projected. The filtered back-projection has both positive (in blue) and negative (in red) components. The more peripheral components are smaller in amplitude. These components occur outside the domain of the original ray. *B*, Each different projection is sequentially filtered and back-projected. This results in a criss-cross pattern of alternating positive and negative components or stripes. *C*, When summation is performed, the effect of the alternating positive and negative components is to cancel out or dampen the spoke pattern associated with simple back-projection, resulting in a better definition and localization of the dense object.

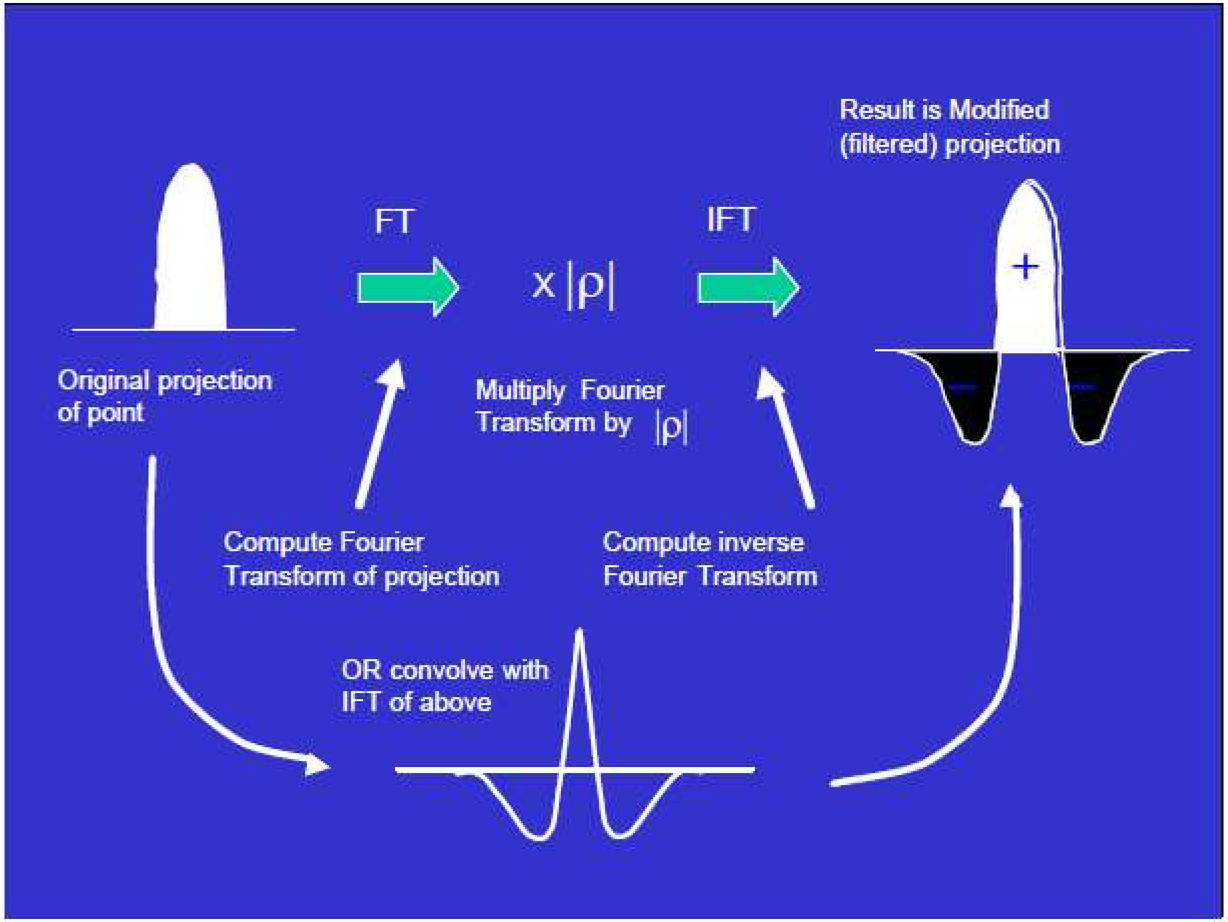
Compare to this unfiltered



For example, after only three projections, the lines would intersect to yield a "star-pattern"……

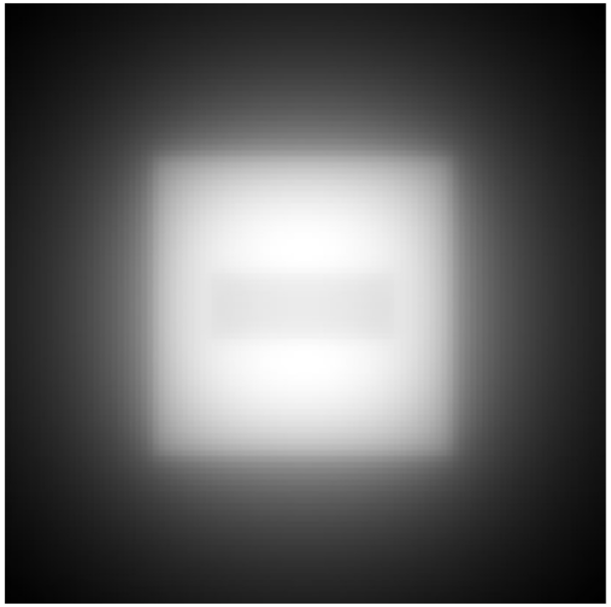


If we now perform the same operation that we performed earlier with the unfiltered projection, we see that the positive parts of the ima ge re-enforce each other, as do the negative components, but that the positive and negative components tend to cancel each other out.

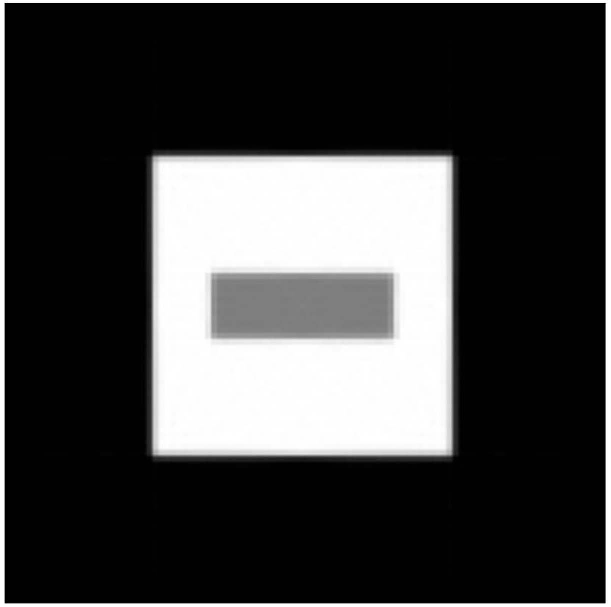


Hence we can take the projection of the cross-section, shown here as a single point, and either perform the processing in the Fourier domain through multiplication with |ρ|, or on the spatial domain by convolving the projection with the IFT of |ρ|. This turns the projection into a "filtered" projection, with negative side-lobes. It is in fact a spatial-frequency-enhanced version of the original projection, with the high-frequency boost being exactly equal to the high-frequency attenuation that is applied during the process of back-projection.

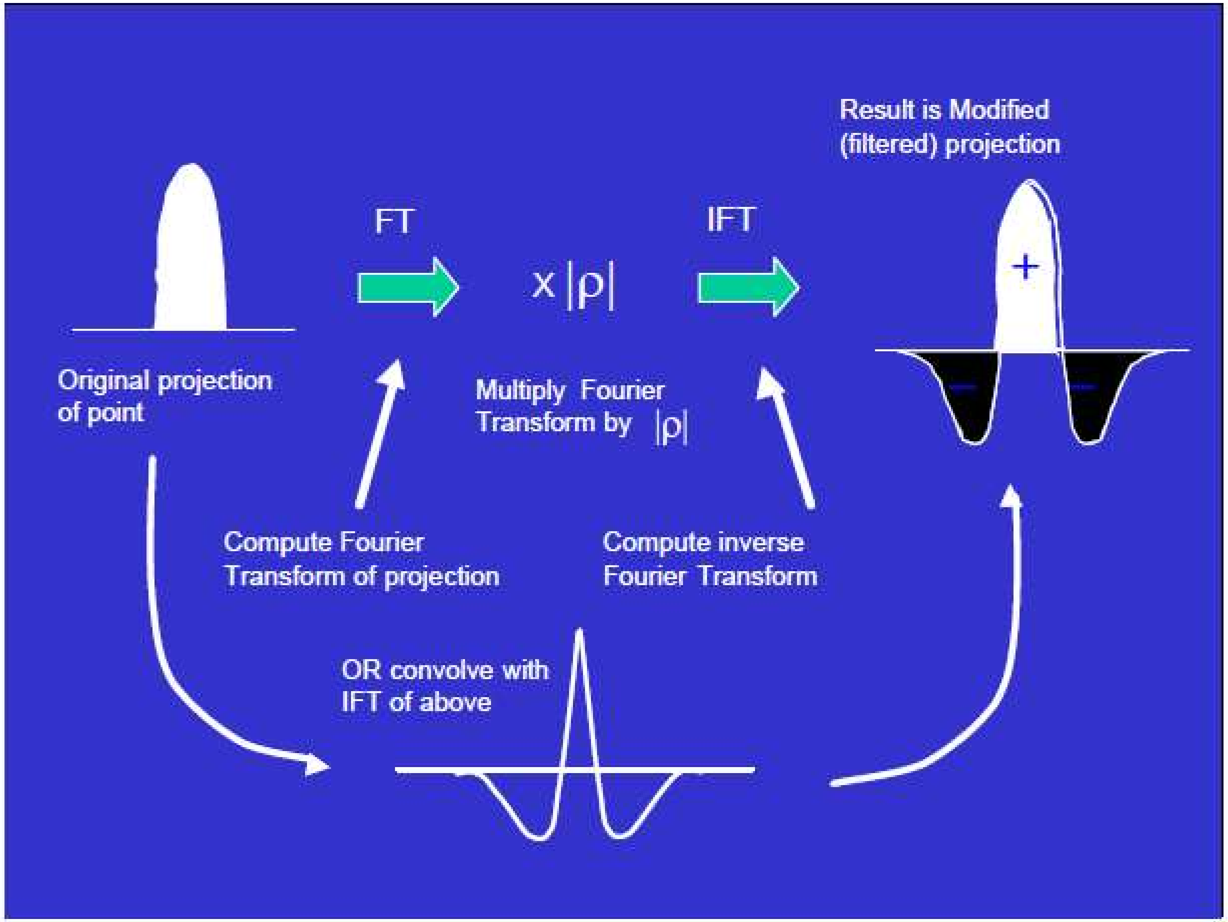
Reconstruction Example



unfiltered



filtered

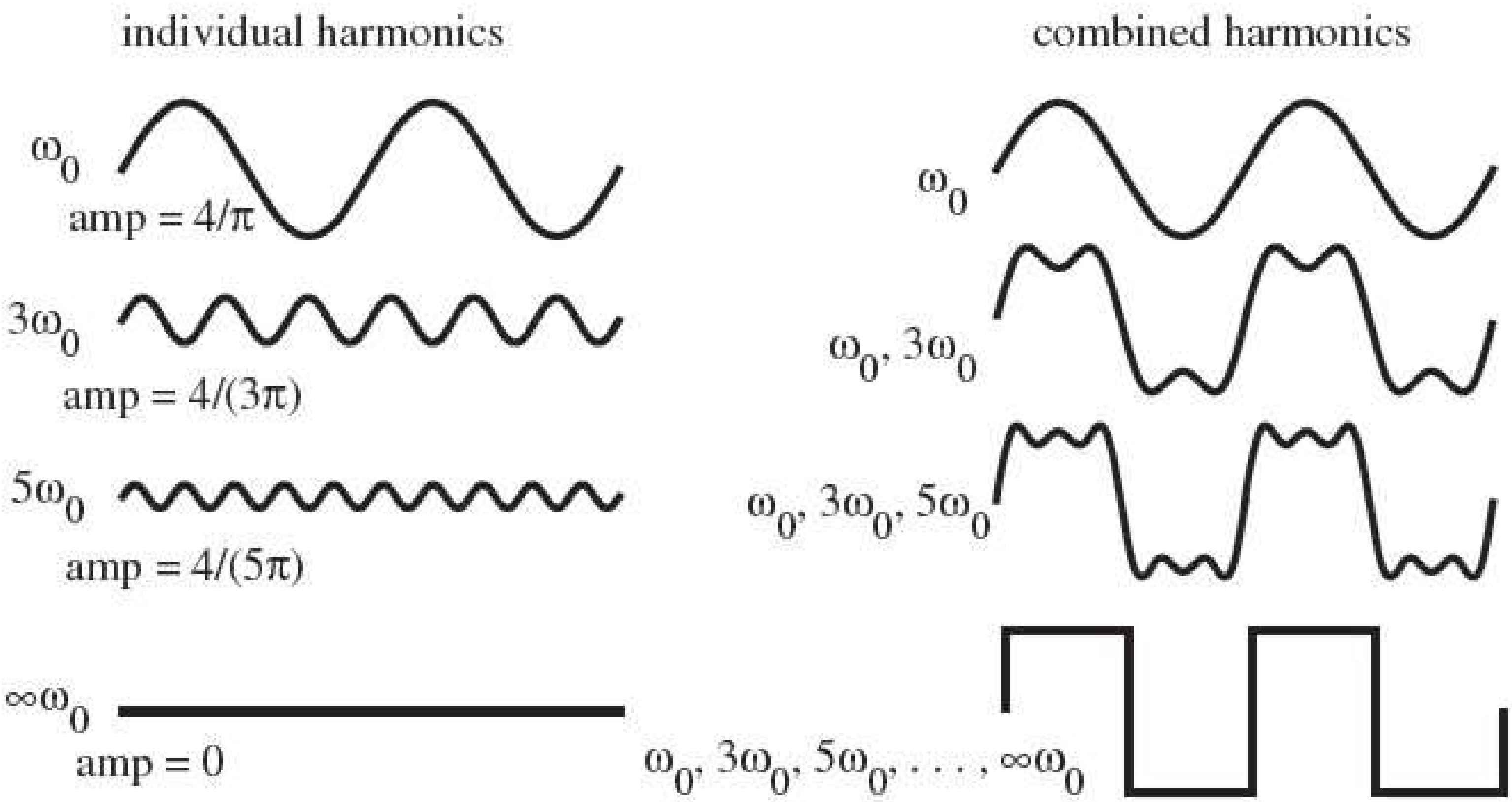


Hence we can take the projection of the cross-section, shown here as a single point, and either perform the processing in the Fourier domain through multiplication with |ρ|, or on the spatial domain by convolving the projection with the IFT of |ρ|. This turns the projection into a “filtered” projection, with negative side-lobes. It is in fact a spatial-frequency-enhanced version of the original projection, with the high-frequency boost being exactly equal to the high-frequency attenuation that is applied during the process of back-projection.

1D Fourier Transformation

Review

1D Fourier Basis



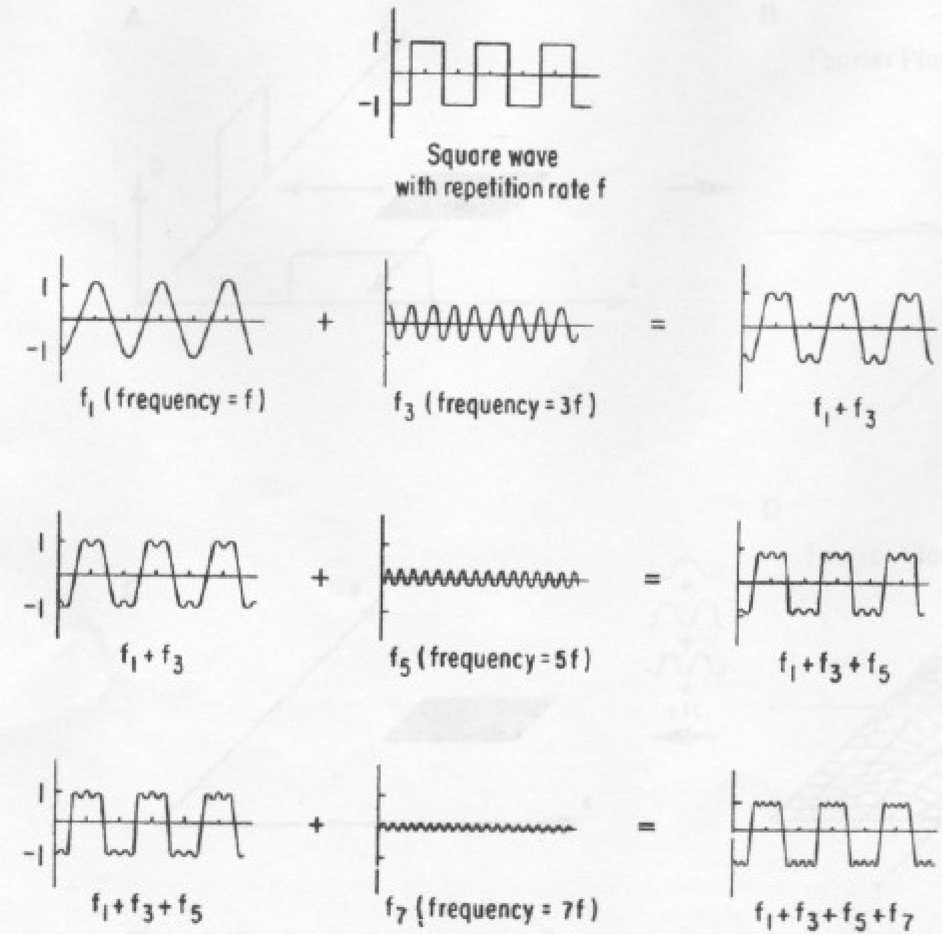
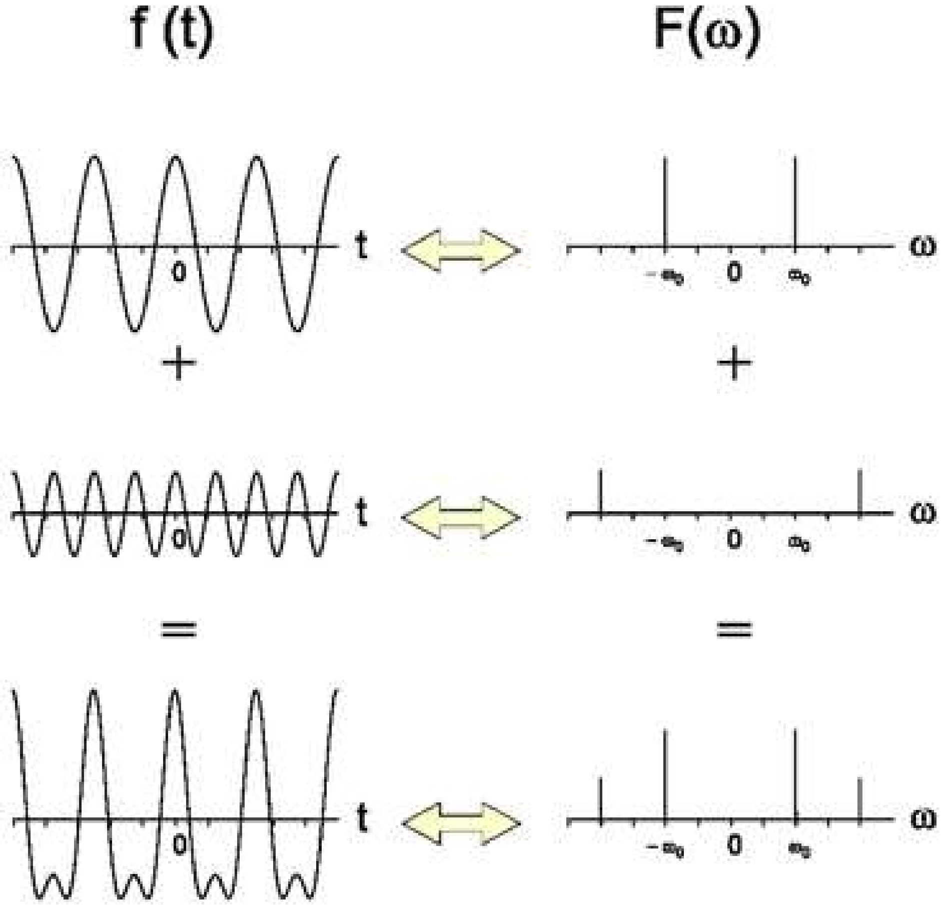


Figure 5.9. Fourier analysis. A square wave pattern illustrates the basic principles of Fourier analysis. By combining sinusoidal waves of the appropriate frequencies and amplitudes, a square wave can be approximated. As more higher frequency components are included, the approximation improves. Any periodically recurring function, or even one that is not periodic or even recurrent, can be synthesized from the appropriate combination of sine waves.

1D Fourier Transform



Filtering

Back to Convolution Properties

$$y(t) = h(t) * x(t) \Leftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

$$y[n] = h[n] * x[n] \Leftrightarrow Y(e^{j\Omega}) = H(e^{j\Omega})X(e^{j\Omega})$$

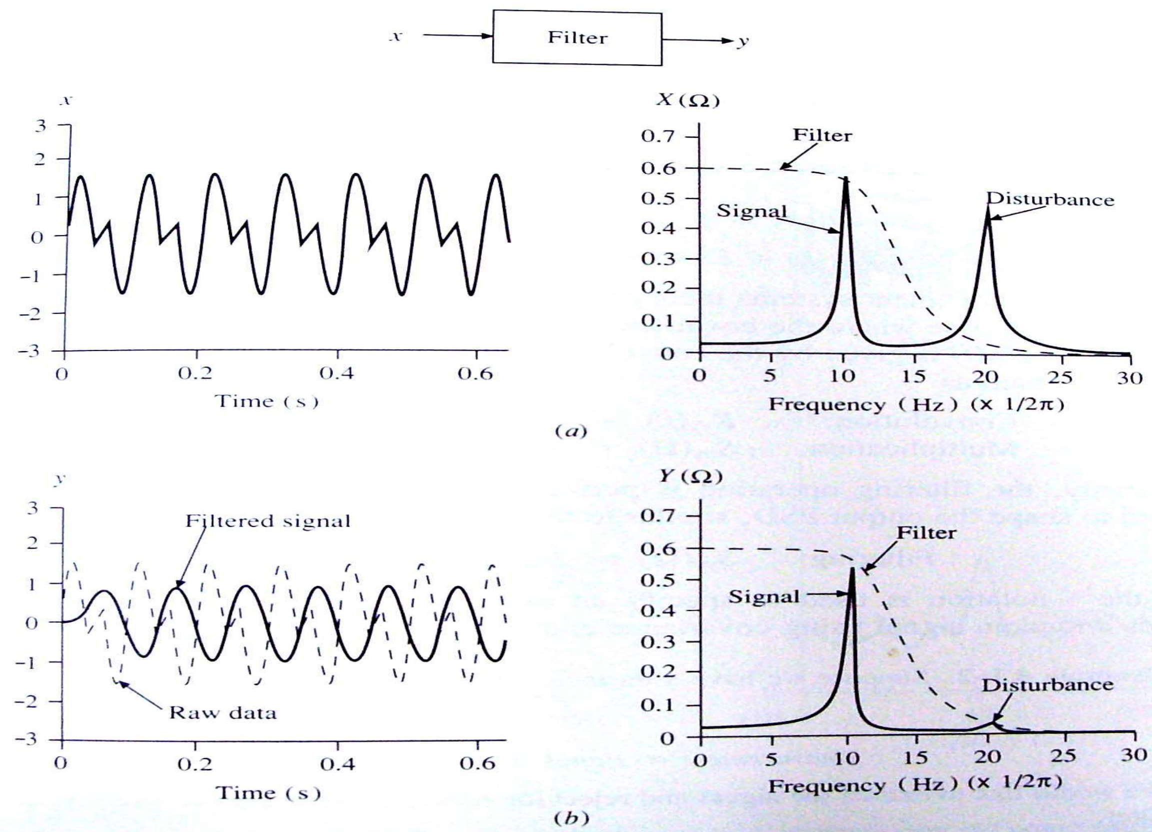
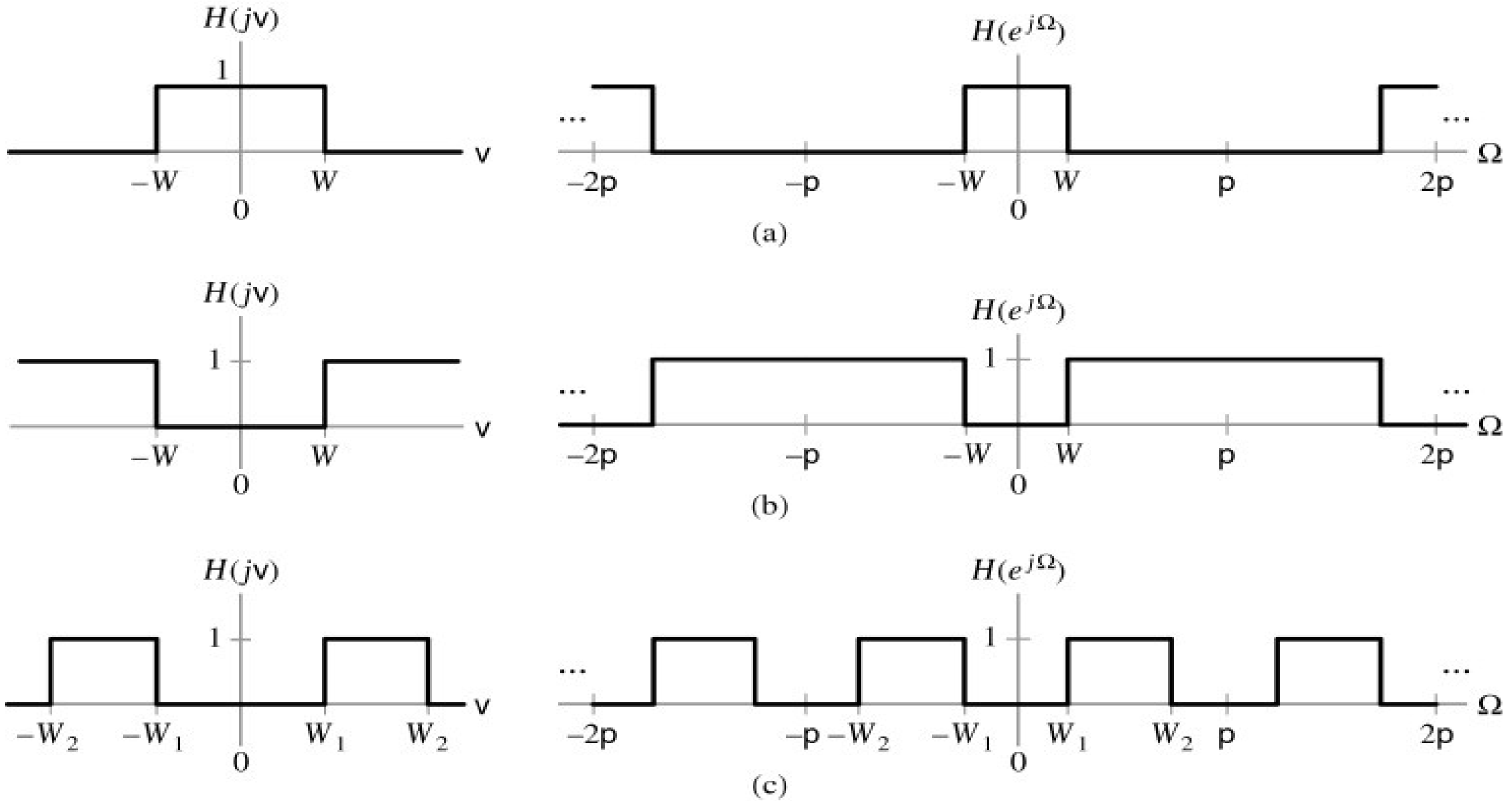


FIGURE 4.1–1
Deterministic signal processing (*a*) Analysis of raw data and spectrum (*b*) Processed data and spectrum

Ideal Filters (1)



Frequency response of **ideal** continuous- (left panel) and discrete-time (right panel) filters. (a) Low-pass characteristic. (b) High-pass characteristic. (c) Band-pass characteristic.

Note the difference between $H(j\omega)$ and $H(j\Omega)$

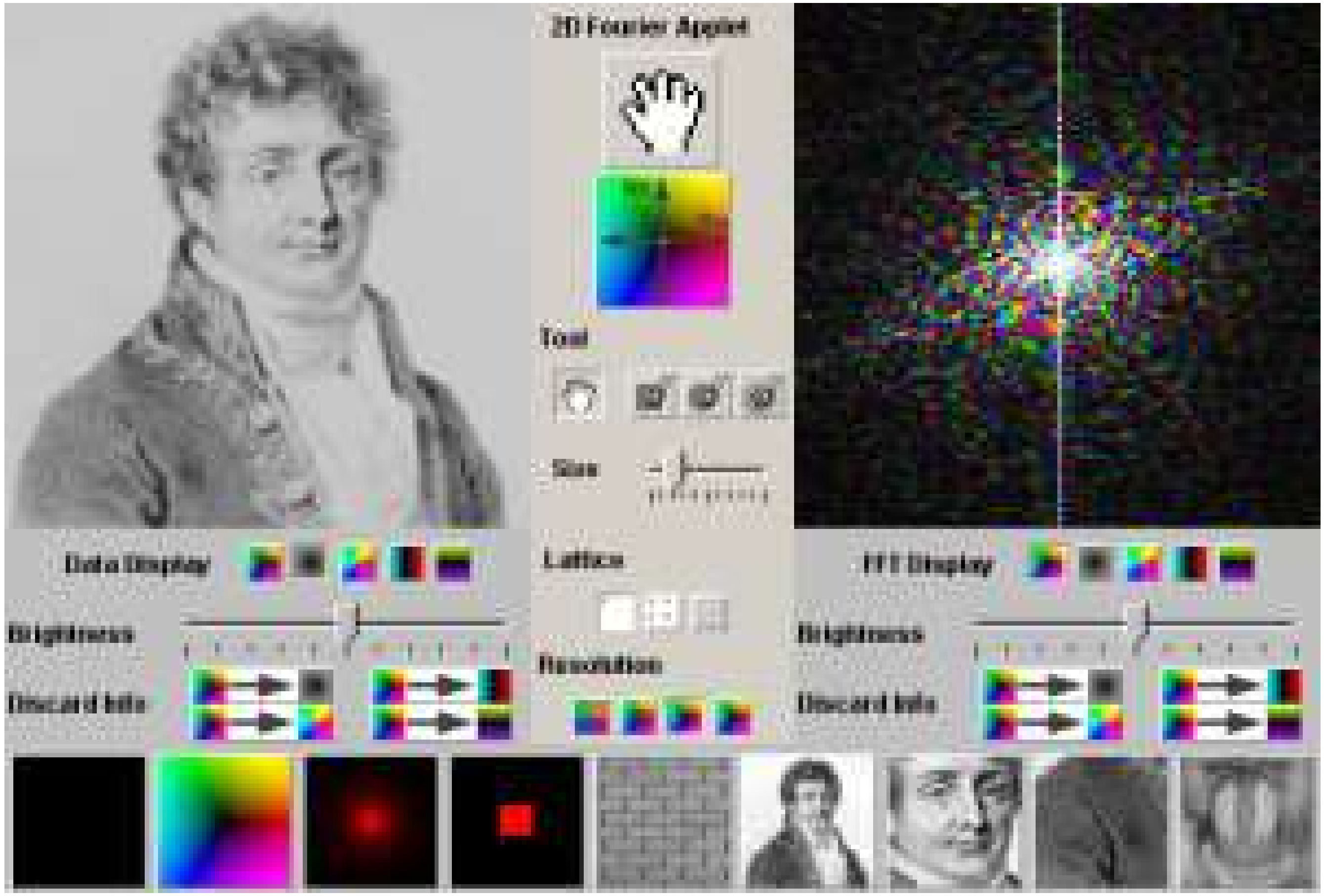
Ideal Filters (2)

- Low-pass filter: filter out high-frequency components of the input. Passes lower frequency components
- High-pass filter: filter out low frequency parts. Passes high frequency parts
- Band-pass filter: passes signals within a certain frequency band

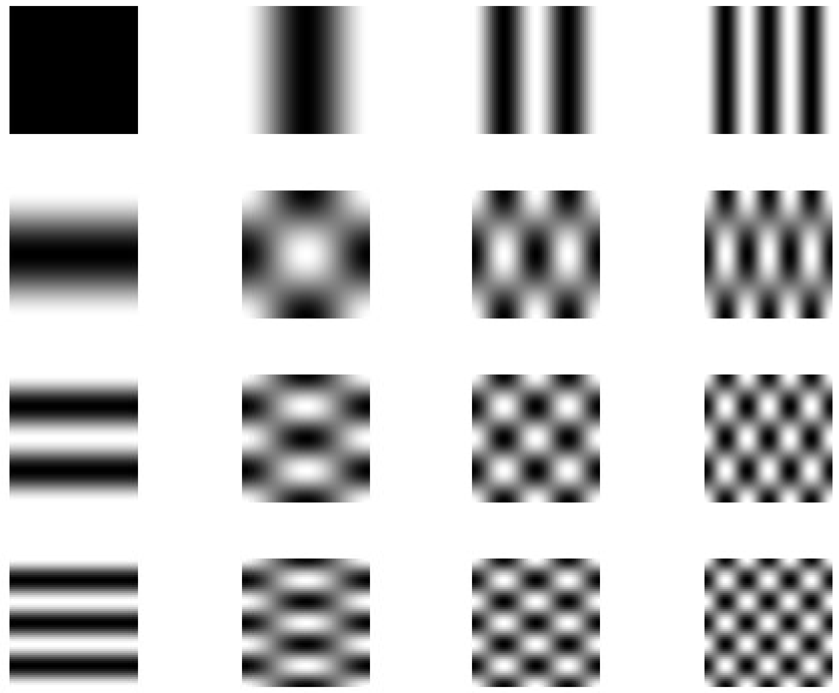
- Pass band=the band of frequencies that are passed by the system
- Stop band=the range of frequencies that are attenuated by the system
- Decibel (dB)=a common unit for the magnitude response of a filter. $dB=20\log|H(j\omega)|$
- -3dB corresponds to 1/sqrt(2) of a magnitude response
- -3dB corresponds to frequencies of which the filter only passes half of the input power.

2D Fourier Transformation

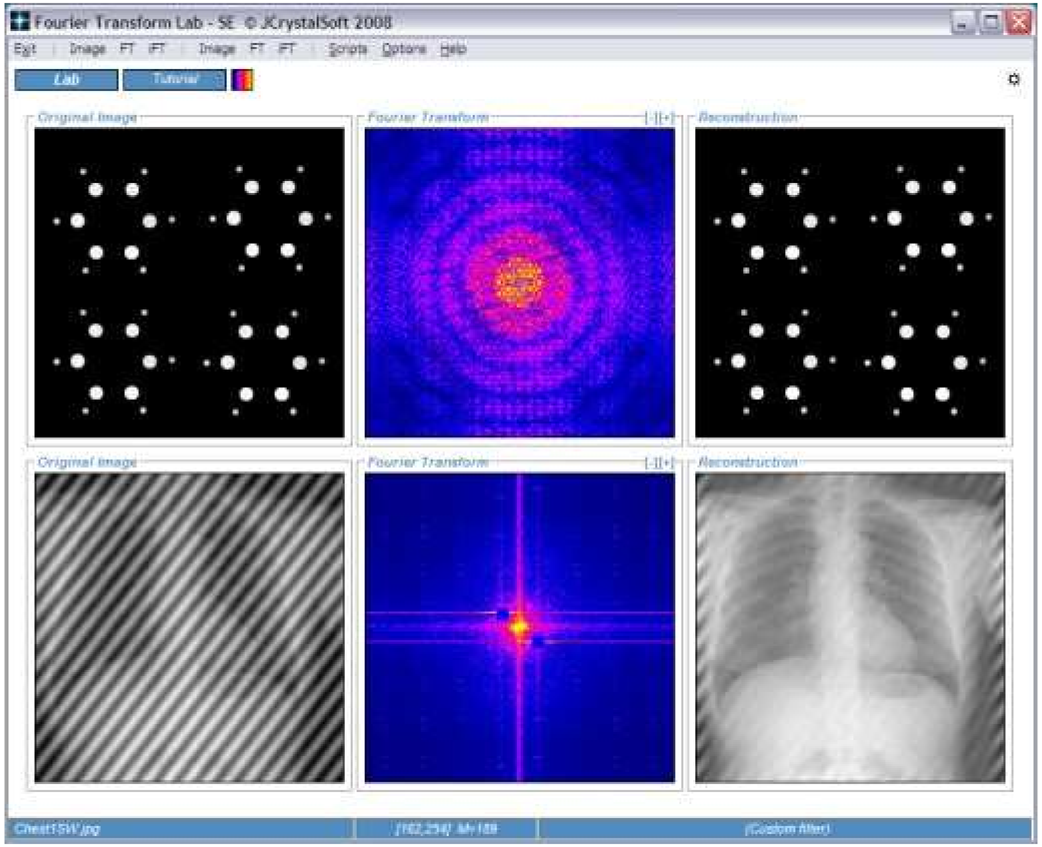
2D Fourier Transform



2-D Fourier Basis



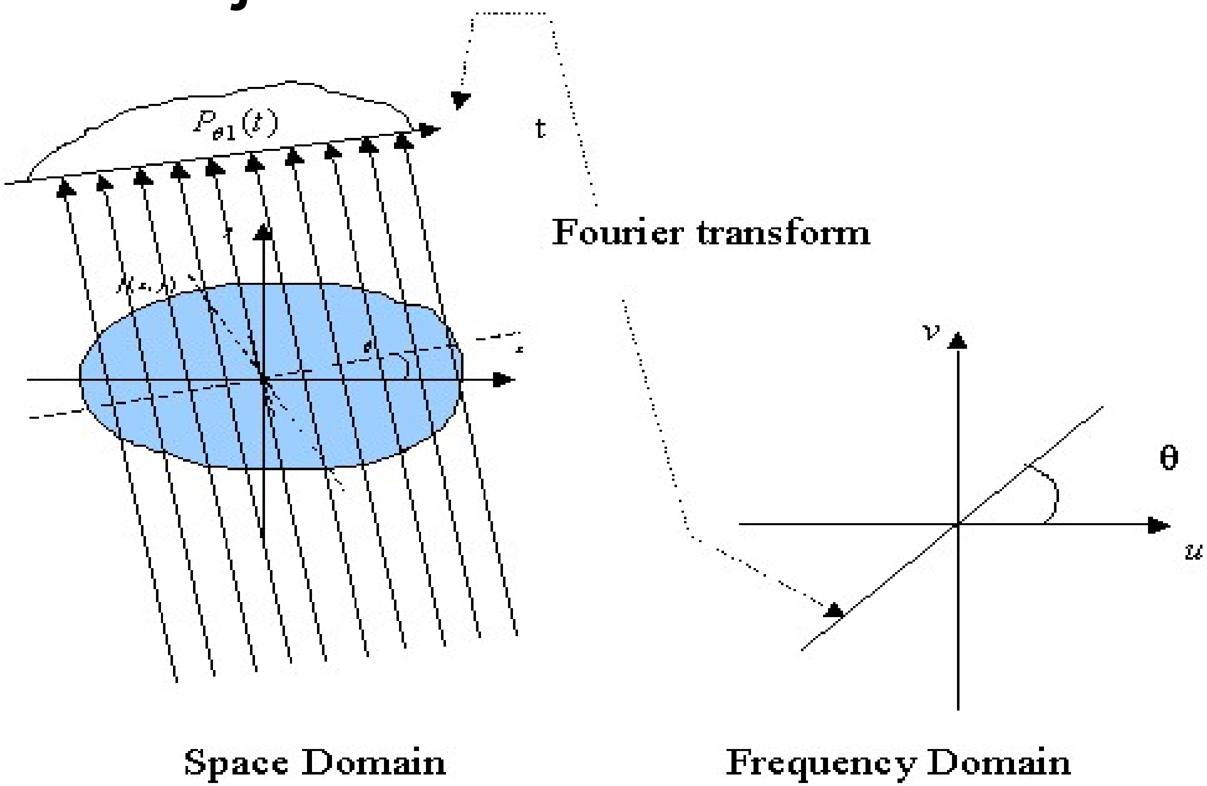
2-D FT & Filtering Example



Fourier Slice Theorem

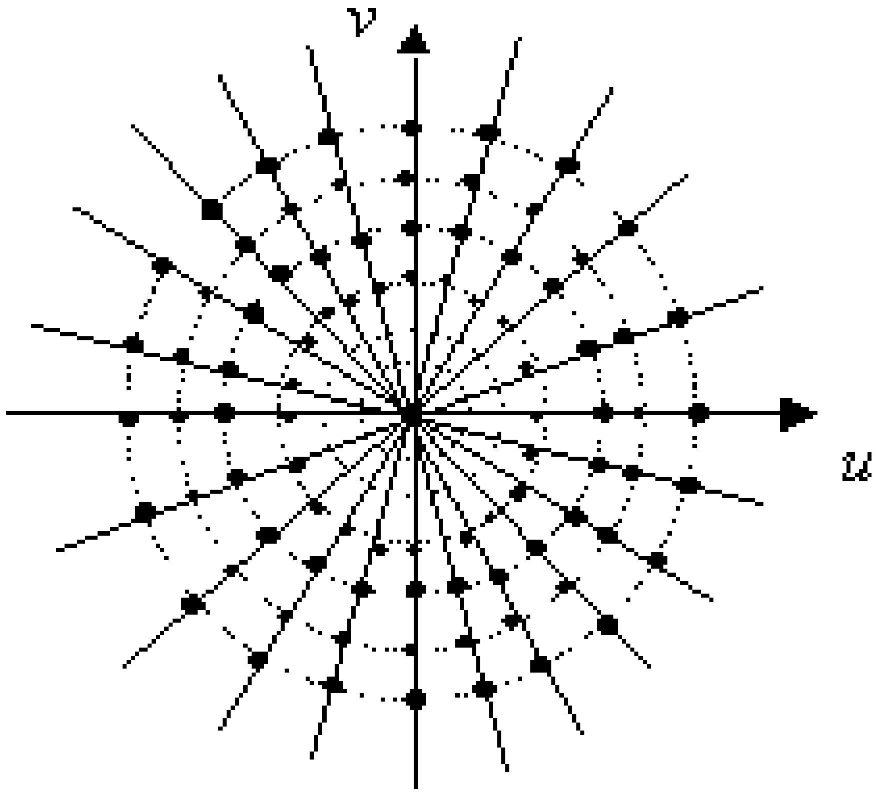
Fourier Slice Theorem

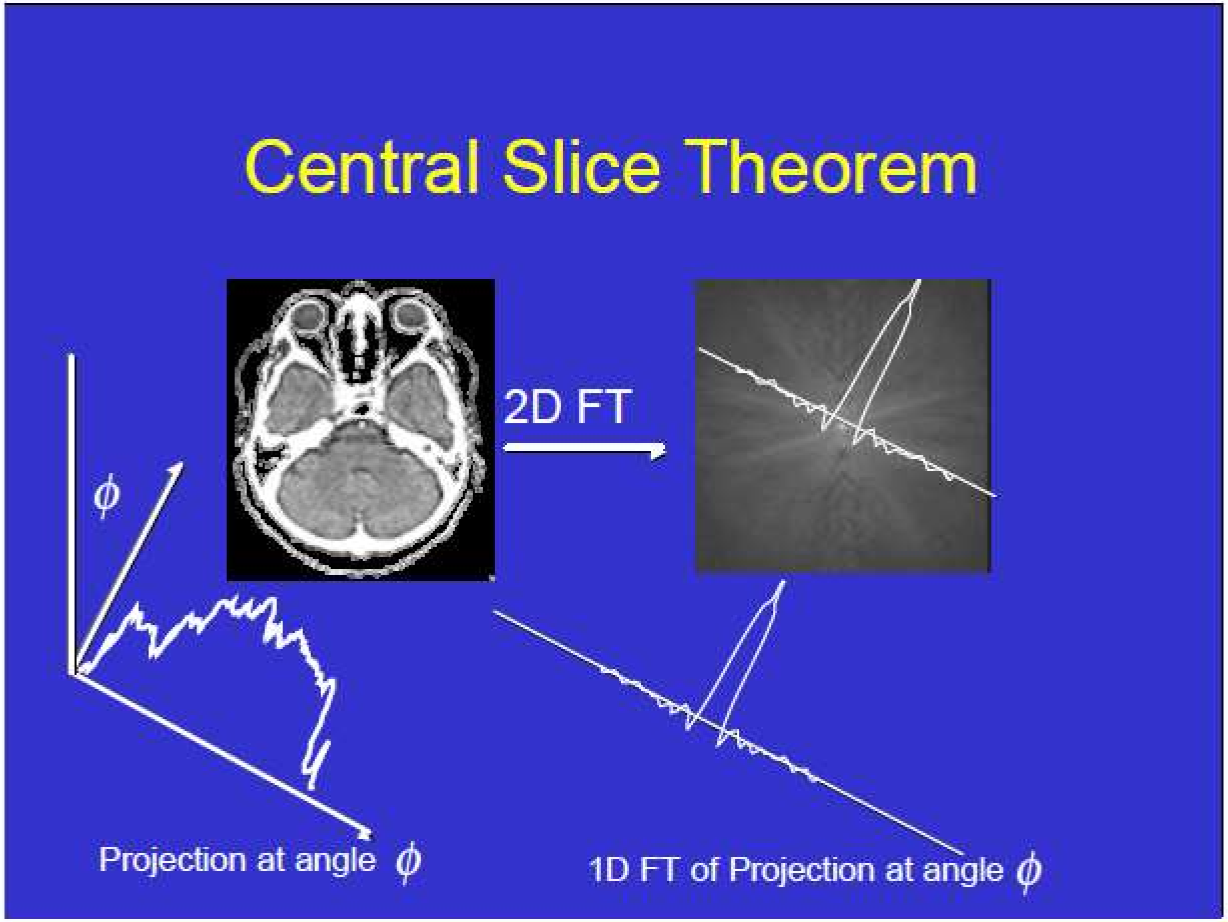
- FT of the projection of a 2-D object is equal to a slice through the origin of 2-D FT of the object.



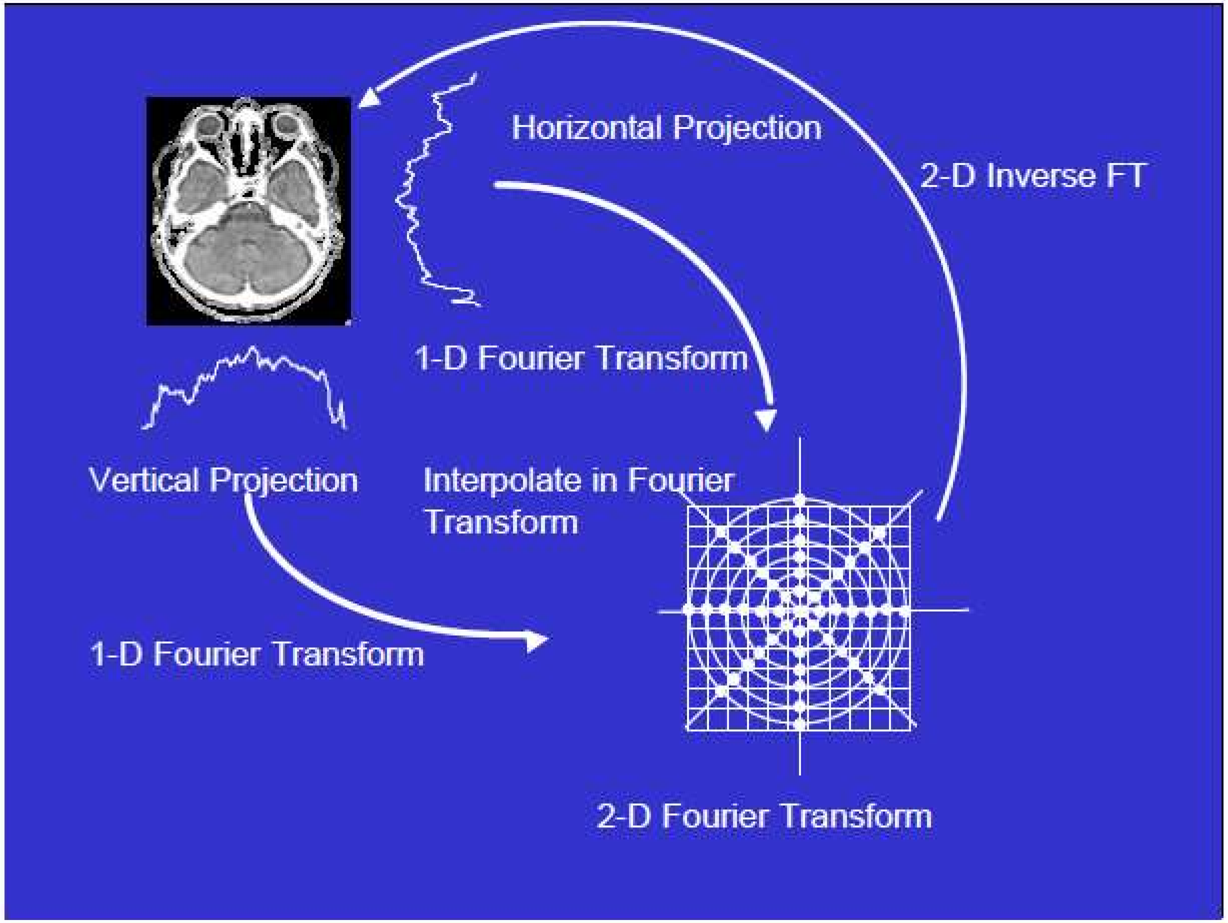
Collection of projections of an object at a number of angles

- In Fourier domain



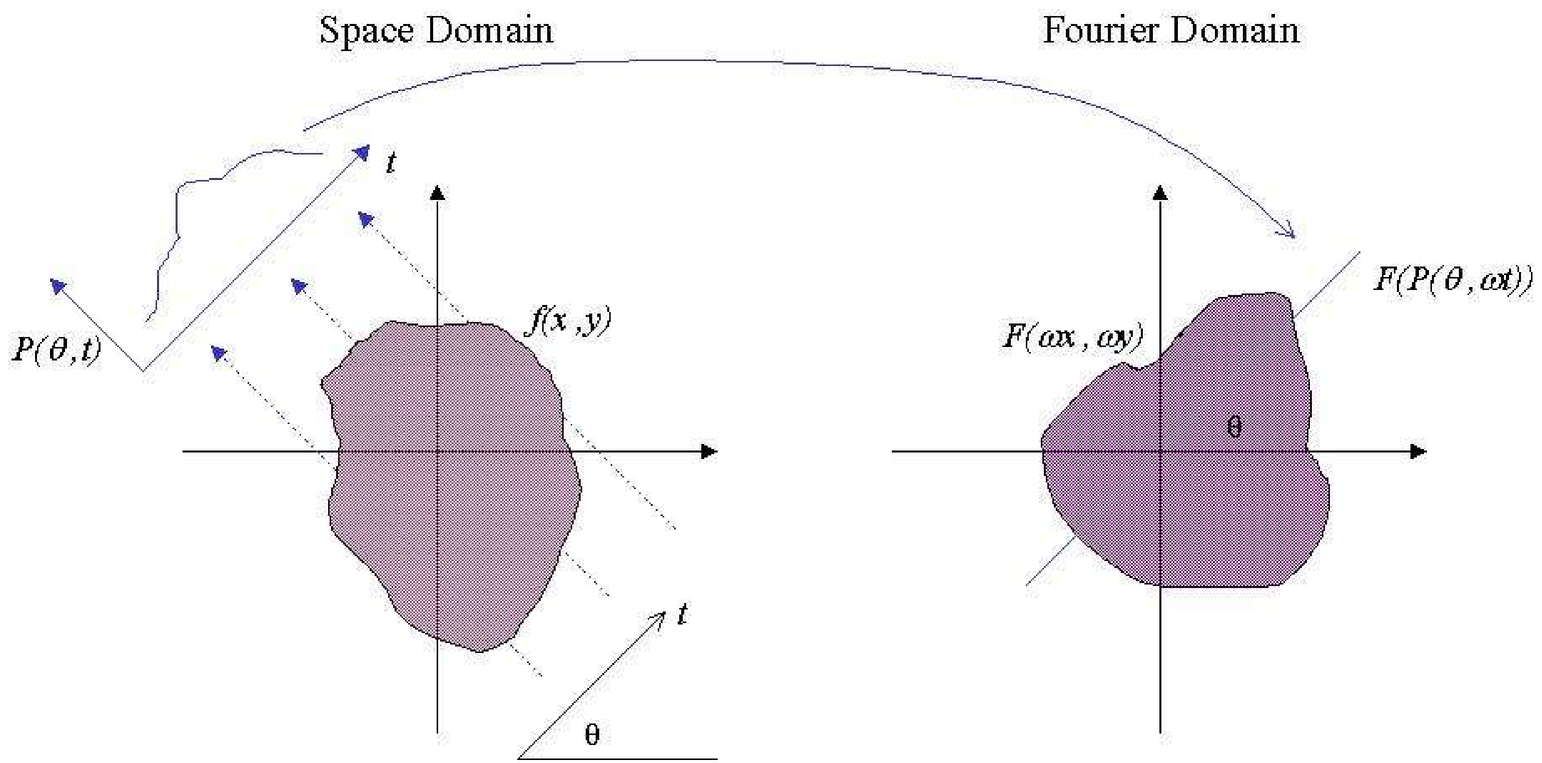


The 1-D projection of the object, measured at angle ϕ, is the same as the profile through the 2D FT of the object, at the same angle. Note that the projection is actually proportional to exp $(-\int u(x)xdx)$ rather than the true projection $\int u(x)xdx$, but the latter value can be obtained by taking the log of the measured value.



If all of the projections of the object are transformed like this, and interpolated into a 2-D Fourier plane, we can reconstruct the full 2-D FT of the object. The object is then reconstructed using a 2-D inverse Fourier Transform.

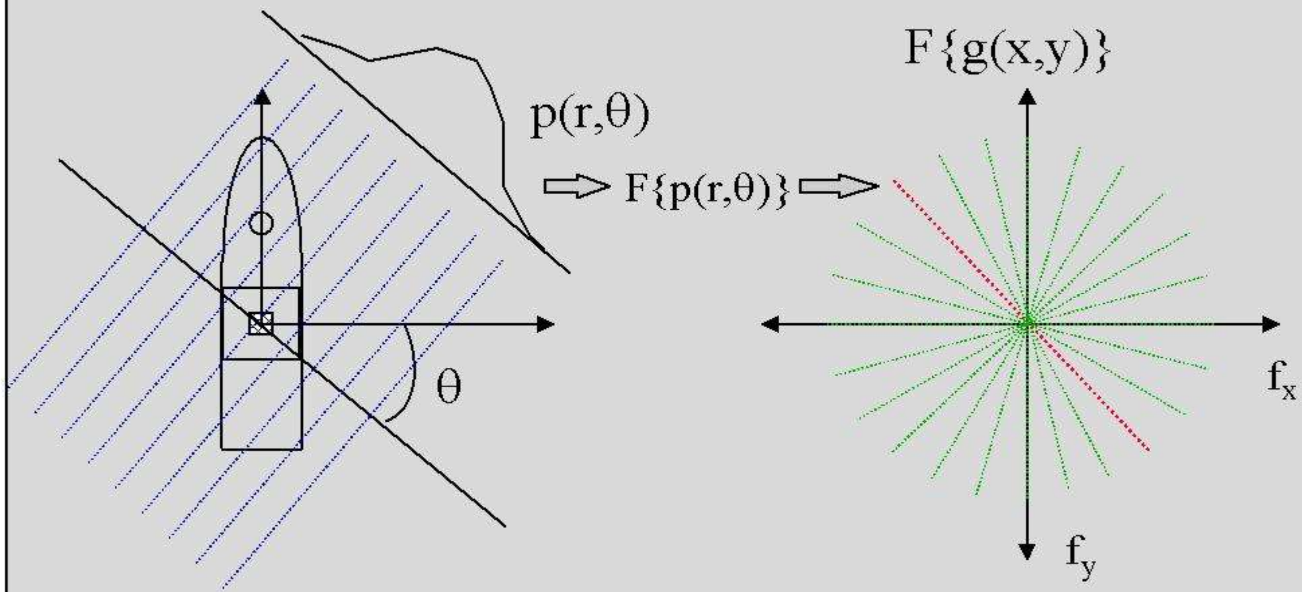
Fourier Slice Theorem



Projection under angle θ equals slice under θ in fourier domain

Tomographic Image Reconstruction

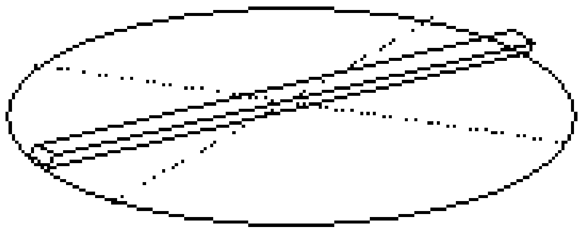
Tomographic Image Reconstruction relies on the Projection Slice Theorem which states that the Fourier transform of image projections at a series of look angles is equivalent to the Fourier transform of the image



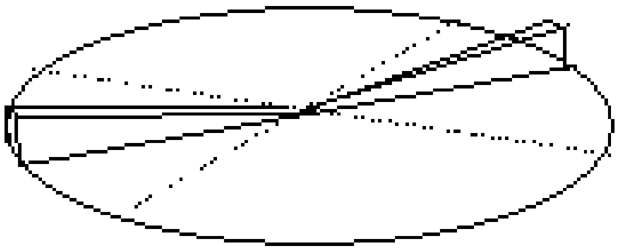
Filtered Back Projection: Fourier Reconstruction

Filtered Projection

- Fourier Slice Theorem



- filtered back projection takes the Fourier Slice and applies a weighting



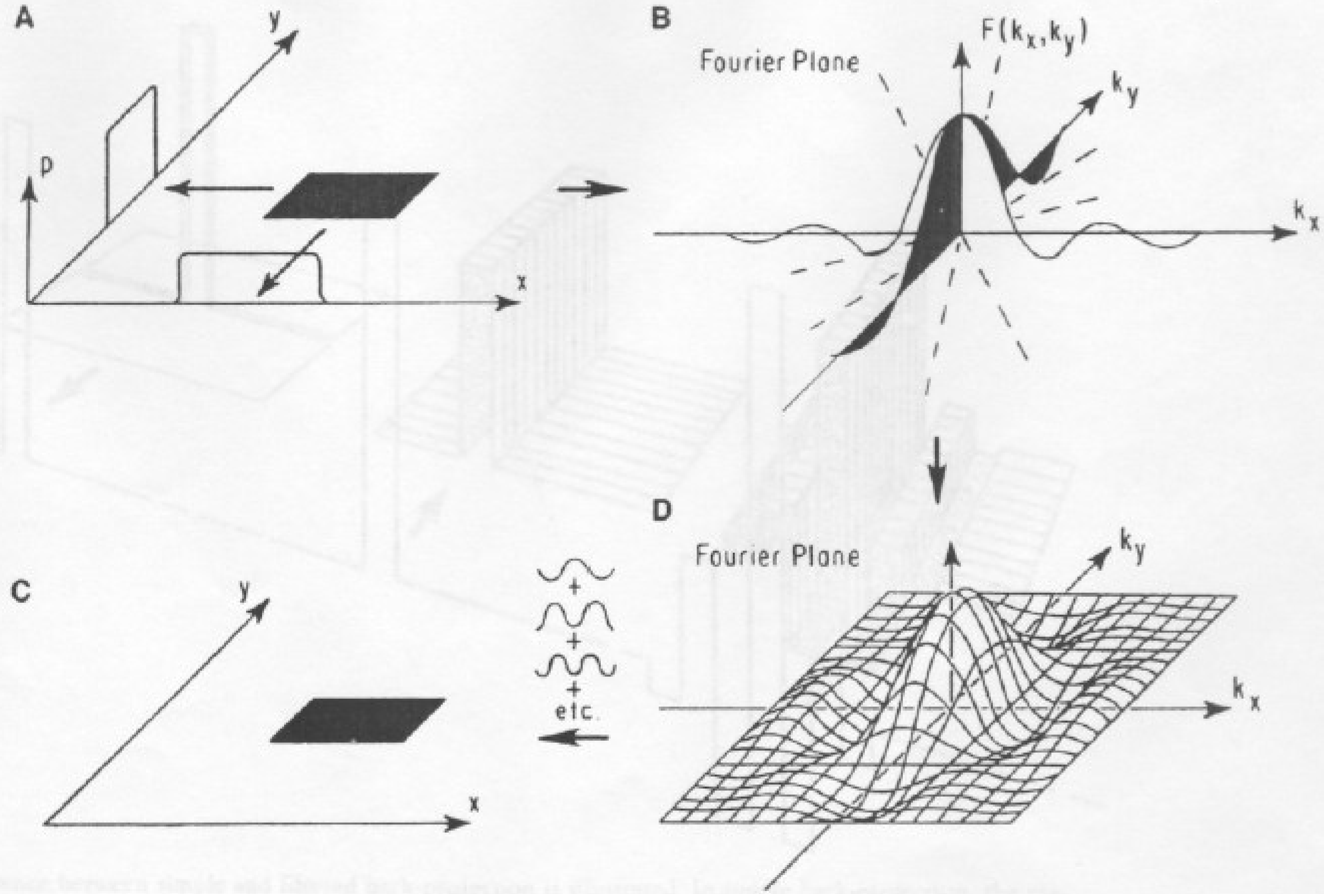
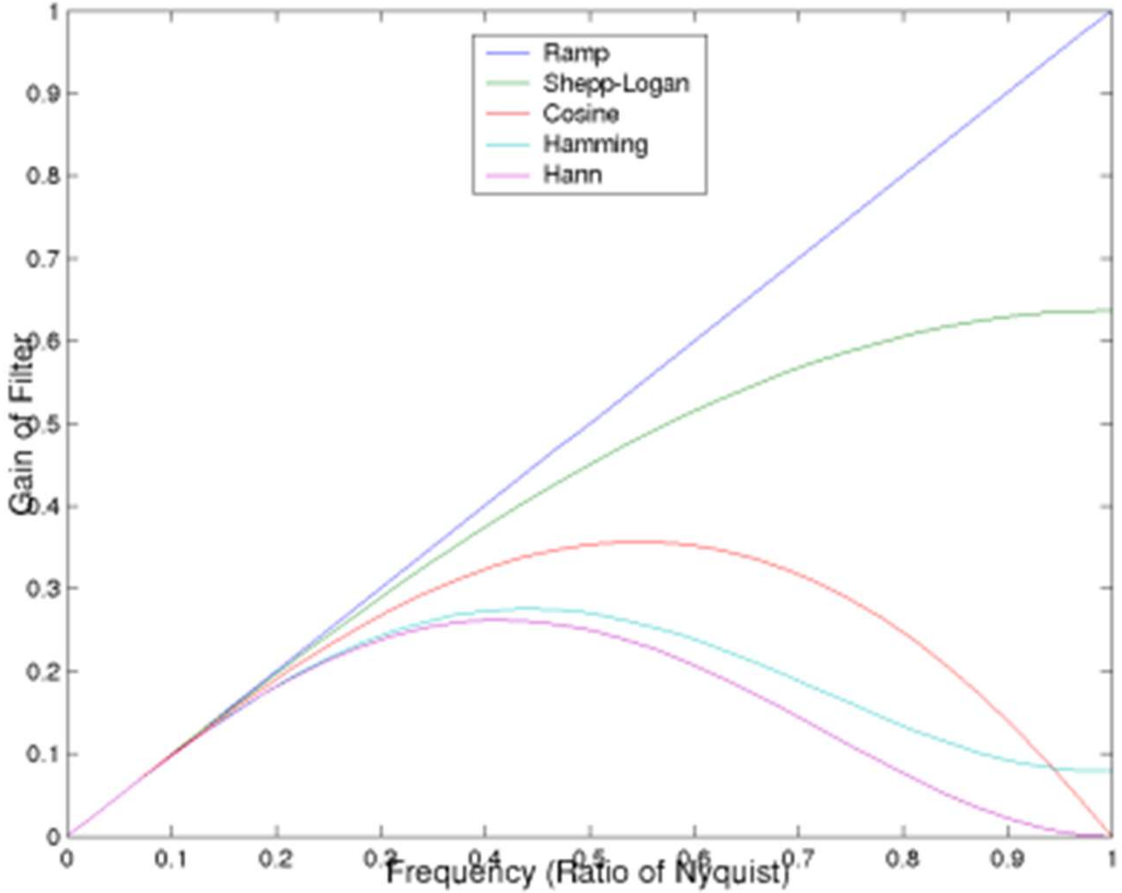
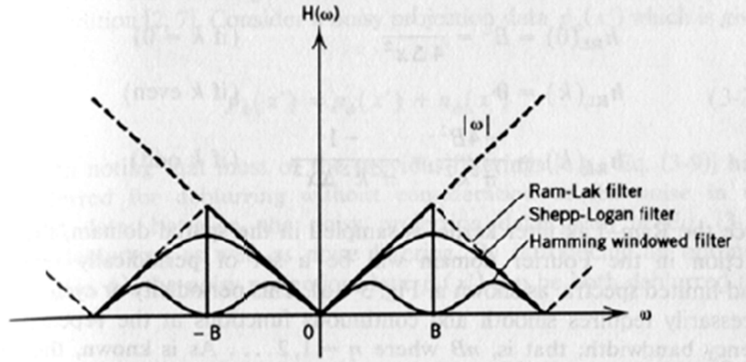


Figure 5.10. Fourier reconstruction. Projections of a rectangular object (*A*) are broken up into sinusoidal waves, whose amplitudes (Fourier coefficients) are plotted along appropriate lines in the frequency plane (*B*) Dashed lines suggest the presence of other transformed projections. A rectangular array of Fourier coefficients is then obtained through interpolation (*C*). Finally, the image is reconstructed by adding together sinusoidal waves (*D*) with amplitudes given by the Fourier coefficients. Thus, *B* and *C* show the amplitudes of sinusoidal waves of various frequencies, which when combined produce the spatial functions *A* and *D*. (Reproduced with permission from: Brooks, R. A., and Di Chiro, G. Theory of image reconstruction in computed tomography. Radiology 117:561–572, 1975).

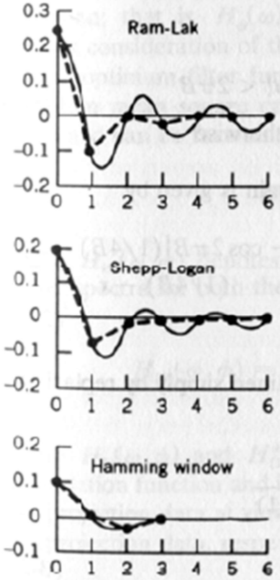
Digital Filters





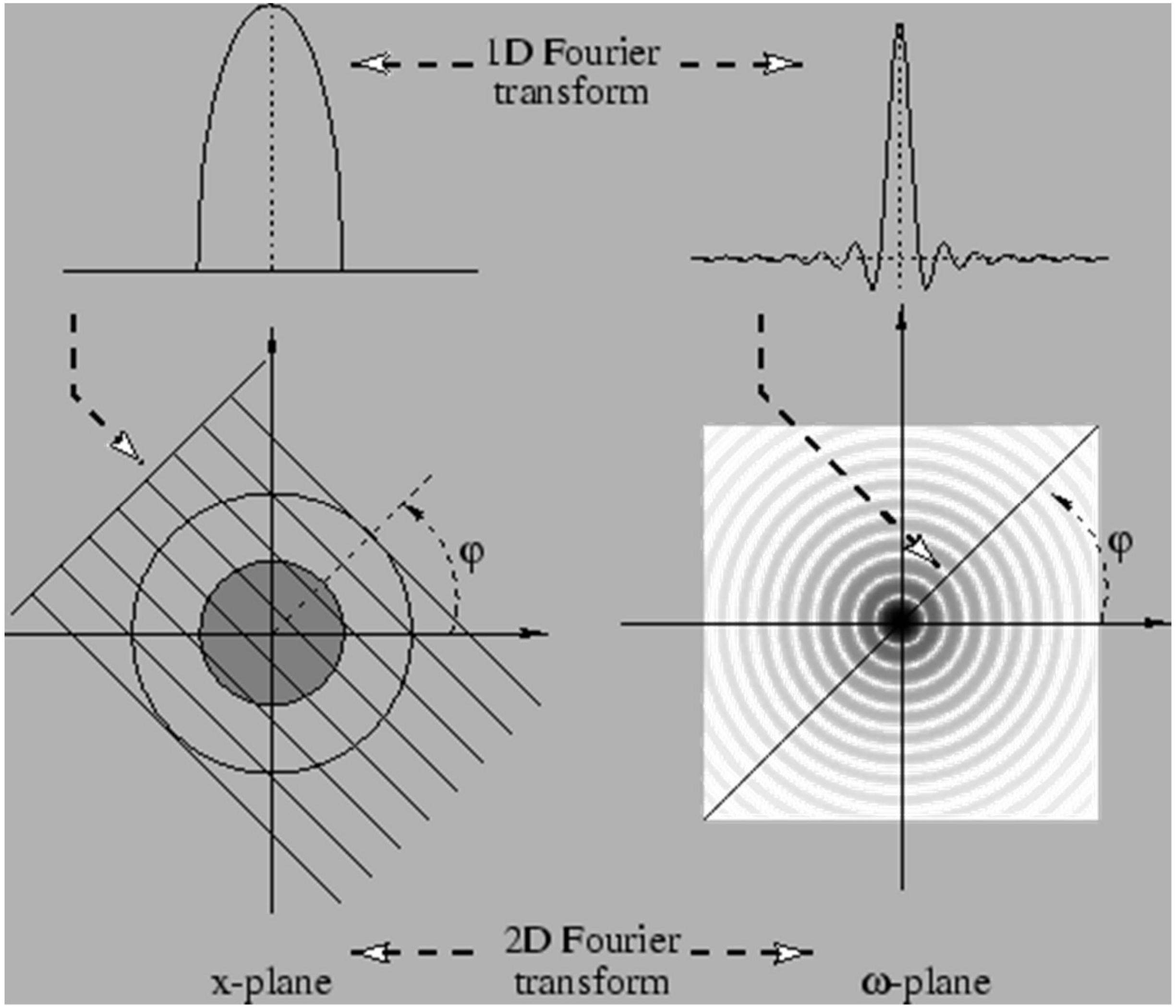
(a)

Figure 3-4 (a) Examples of the band-limited filter function of sampled data. Note the cyclic repetitiveness of the digital filter.



(b)

Figure 3-4 (b) Spatial domain filter kernals corresponding to the filter functions shown in the Ram-Lak filter is a high-pass filter with a sharp response but results in some noise enhancement, while the Shepp-Logan and the Hamming window filters are noise-smoothed filters and therefore have better SNR.


1D Fourier transform
φ
φ
2D Fourier transform
x-plane
ω-plane

Filtered Backprojection Algorithm

1. In Matlab, implemented as iradon.m
2. 1-D FFT
3. Digital Filters
4. Interpolation Functions
5. 2-D Inverse FFT

Approaches to Backprojection