Analysis of Hodgkin-Huxley Equations

$$C\dot{v} + g(v - E) = 0$$

$$g = g_{Na} + g_k + g_{Cl}$$

$$E = \frac{g_{Na}E_{Na} + g_K E_K + g_{Cl}E_{Cl}}{g_{Na} + g_K + g_{Cl}}$$

g = total membrane conductance

E = weighted average of the equilibrium potentials

- Membrane potential v is always approaching the instantaneous value of E
- When $v < E$, we have $\dot{v} > 0$. When $v > E$, we have $\dot{v} < 0$. Therefore the cell can adjust E simply by changing the membrane conductance
- In the cell at rest, v and E are both close to E_{Cl}.
- Early in an action potential, Na^+ channels open, dramatically increase g_{Na}. This makes E become close to E_{Na} and v follows
- Later in the action potential, the Na^+ channels close and K^+ channels open. This brings E become close to E_K . thus the membrane potential v becomes more negative than the resting potential of the cell
- Finally, all of the conductances come back to normal, and E and v return to their resting values, close to E_{Cl}.

$g_K = \bar{g}_K n^4$

$$\frac{dn}{dt} = \alpha_n(v)(1-n) - \beta_n(v)n$$

These equations are based on the following postulates

1. Each K^+ channel has four gates. Each gate can be open or closed. The K^+ channel as a whole is open if an only if all four of its gates are open.
2. All four gates within a K^+ channel are identical
3. The different gates within a K^+ channel operate independently of one another
4. The rate constant (probability per unit time) for opening or closing a gate of a K^+ channel is a specified function of voltage for the opening rate and another specified function of voltage for the closing rate.

- To specify the conductance g_K, consider there are four independent gates in each channel and since n is the probability that each of them is open, then the probability that all four are open is equal to n^4
- Let $n(t)$ be the fraction of K^+ gates that are in the open state at time t
- $\alpha_n(v)$ is the opening rate constant for the K^+ gates which is a function of voltage. Also it is an increasing function
- $\beta_n(v)$ is the closing rate constant for the K^+ gates which is a function of voltage. It's a decreasing function
- Increasing the membrane potential encourages K^+ gates to open and discourages them from closing, thus increasing the K^+ conductance.
- At any particular v, $\alpha_n(v)$ and $\beta_n(v)$ are just number.
- The opening rate constant $\alpha_n(v)$ is multiplied by *(1-n).* Because *(1-n)* is the fraction of closed gates, and a gate has to be closed in order to open.
- If *n=1,* all of the gates are already open, so the rate of opening has to be zero at that particular instant. Similarly, the closing rate constant $\beta_n(v)$ is multiplied by *n* in equation because *n* is the fraction of open K^+ gates, and a gate has to be open in order to close.

$g_{Na} = \bar{g}_{Na} m^3 h$
$\dot{m} = \alpha_m(v)(1-m) - \beta_m(v)m$
$\dot{h} = \alpha_h(v)(1-h) - \beta_h(v)h$

The postulates

1. Each Na^+ channel has four gates. Each gate can be open or closed.
2. The four gates are not identical. Three of them are of a type that we shall call an m-gate. One is of a type that we shall call an h-gate.
3. The different gates within a Na^+ channel operate independently of one another, regardless of their type.
4. The rate constant for opening or closing a gate of a Na^+ channel is a specified function of voltage for the opening rate and another specified function of voltage for the closing gate.

- To specify the Na^+ conductance, g_{Na} as a function of *m* and *h*. Since there are three m-gates and one h-gate in each channel and different channels operate independently, the probability that all four gates are open is equal to $m^3 h$
- Let $m(t)$ be the fraction of open m-gates at time t, and let $h(t)$ be the fraction of open h-gates at that time.
- The functions $\alpha_m(v), \beta_m(v), \alpha_h(v), \beta_h(v)$ give the opening and closing rate constants for the m-gates and the h-gates
- $\alpha_m(v), \beta_h(v)$ are increasing functions
- $\beta_m(v), \alpha_h(v)$ are decreasing functions
- Increasing voltages encourages opening and discourages closures of the m-gates, but has just the opposite effect on the h-gates.
- The m-gates respond to voltage changes about 10x faster than the h-gates

$g_L = \bar{g}_L$

- A third type of channel in the membrane. It is called a leakage channel
- The leakage channel population has a constant conductance.

$\alpha_n(v), \beta_n(v), \alpha_m(v), \beta_m(v), \alpha_h(v), \beta_h(v)$

- Expression for these rate constants are found by curve fitting of the above equations to the measure data.

- Their expressions are given as functions of voltage (3.5.25-3.5.30)

$$\alpha_{\mathrm{m}}(v) = 1.0\frac{(v+45)/10}{1-\exp\left(-(v+45)/10\right)}, \quad (3.5.25)$$

$$\beta_{\mathrm{m}}(v) = 4.0\exp\left(-(v+70)/18\right), \quad (3.5.26)$$

$$\alpha_{\mathrm{h}}(v) = 0.07\exp\left(-(v+70)/20\right), \quad (3.5.27)$$

$$\beta_{\mathrm{h}}(v) = 1.0\frac{1}{1+\exp\left(-(v+40)/10\right)}, \quad (3.5.28)$$

$$\alpha_{\mathrm{n}}(v) = 0.1\frac{(v+60)/10}{1-\exp\left(-(v+60)/10\right)}, \quad (3.5.29)$$

$$\beta_{\mathrm{n}}(v) = 0.125\exp\left(-\frac{v+70}{80}\right). \quad (3.5.30)$$

- Constants in the H-H equations (3.5.31-3.5.37)

$$C = 1.0\,\frac{\text{microamperes} \times \text{milliseconds}}{\text{centimeter}^2}, \quad (3.5.31)$$

$$\bar{g}_{\mathrm{Na}} = 120\,\frac{\text{microamperes/millivolt}}{\text{centimeter}^2}, \quad (3.5.32)$$

$$\bar{g}_{\mathrm{K}} = 36\,\frac{\text{microamperes/millivolt}}{\text{centimeter}^2}, \quad (3.5.33)$$

$$\bar{g}_{\mathrm{L}} = 0.3\,\frac{\text{microamperes/millivolt}}{\text{centimeter}^2}, \quad (3.5.34)$$

$$E_{\mathrm{Na}} = 45\,\text{millivolts}, \quad (3.5.35)$$

$$E_{\mathrm{K}} = -82\,\text{millivolts}, \quad (3.5.36)$$

$$E_{\mathrm{L}} = -59\,\text{millivolts}. \quad (3.5.37)$$