The Heart & Circulation (III)

1.11 Dynamics of the Arterial Pulse

- Fig. 1.10: systemic arterial blood pressure (P) as a function of time (t).
- Fig. 1.15: Model of the left heart and systemic arteries.

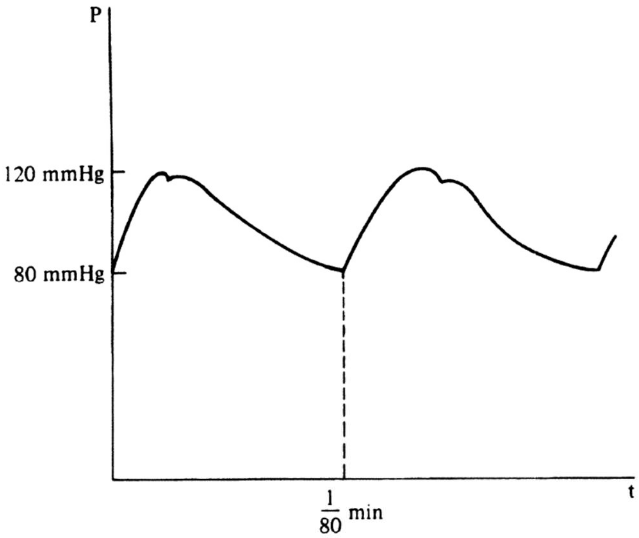

Figure 1.10. Systemic arterial blood pressure (P) as a function of time (t). The normal maximum (systolic) and minimum (diastolic) pressures are indicated on the pressure axis, and a typical period of the heartbeat is indicated on the time axis.

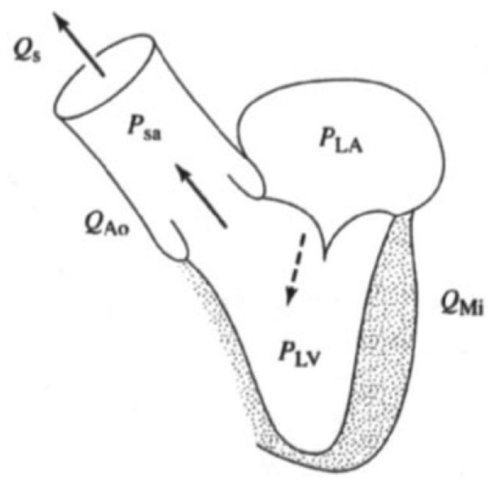

Figure 1.15. Model of the left heart and systemic arteries. Pressures and flows are P_{LA} = left atrial pressure, P_{LV} = left ventricular pressure, P_{sa} = systemic arterical pressure, Q_{Mi} = flow through the mitral valve (indicated here by a dotted arrow because the mitral valve is closed at the moment shown), Q_{Ao} = flow through the aortic valve, and Q_s = outflow from the systemic arterial tree, i.e., aggregate flow through all of the tissues of the body.

- Let's build a simple model
- Assume a compliance vessel, but not in the steady state (see Fig. 1.11)

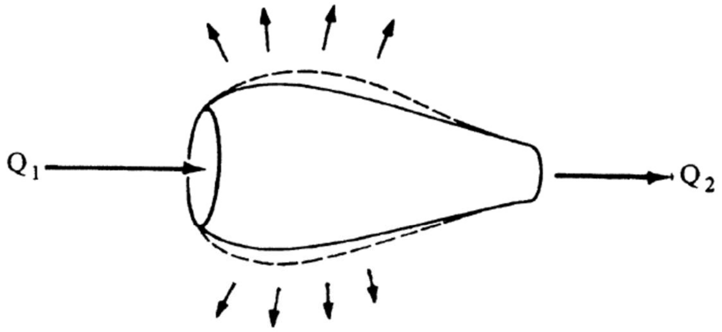

Figure 1.11. Volume conservation in pulsatile flow. The rate of change of volume of a vessel is equal to its inflow (Q_1) minus its outflow (Q_2).

- The change in volume of vessel = the rate of change of the vessel's volume is the difference between the inflow and the outflow.

$$\dot{V} = \frac{dV}{dt} = Q_1 - Q_2$$

If $Q_1 = Q_2$, then steady-state.
where $Q_1(t) = Q_1$= inflow, $Q_2(t) = Q_2$= outflow

- For a compliance vessel

$$\mathrm{V(t) = CP(t)} \text{ or } \mathrm{V(t) = CP(t)} + V_d$$
$$\dot{V} = C\dot{P} = Q_1 - Q_2$$

- Introduce $\mathrm{P} = P_{sa}$, $\mathrm{C} = C_{sa}$,
  output of the left heart $Q_1 = Q_L$,
  blood flow through the systemic tissue $Q_2 = Q_S$,
  $Q_s = \frac{(P_{sa} - P_{sv})}{R_s} \approx \frac{P_{sa}}{R_s}$ since $P_{sv} \ll P_{sa}$.

- Now we have

$$C_{sa}\dot{P}_{sa} = Q_L(t) - \frac{P_{sa}}{R_s} \qquad (*)$$

- During diastole, when the aortic value is closed, $Q_L = 0$,

$$P_{sa}(t) = P_{sa}(0)\exp(\frac{-t}{R_s C_{sa}}) \qquad (**)$$

This is the solution for (*)

- From $\mathrm{V(t) = CP(t)} + V_d$, a sudden change in volume of magnitude $\Delta V_0 = C_{sa}\Delta P_{sa}$.
- During the duration of the each heartbeat, T and assume the heart beat is periodic 0~T.
  $\Delta P_{sa} = P_{sa}(0) - P_{sa}(T)$ = Systolic Arterial Pressure – Diastolic Arterial Pressure
  (see Fig. 1.12)

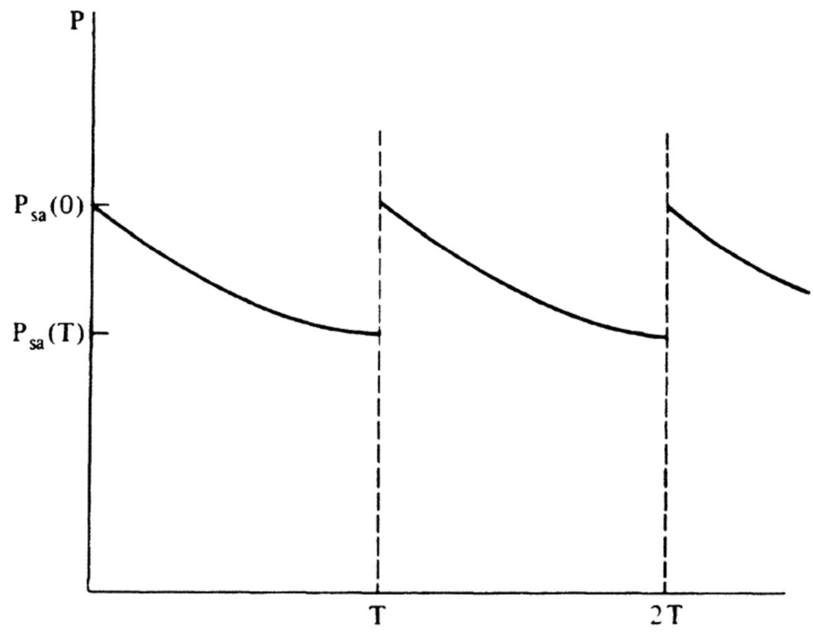

Figure 1.12. Idealized arterial pulse, under the assumption that ejection of the entire stroke volume from the left ventricle occurs instantaneously. T = period of the heartbeat, $P_{sa}(0)$ = systolic pressure, $P_{sa}(T)$ = diastolic pressure.

- t → T in (**) $$P_{sa}(T) = P_{sa}(0)\exp\left(\frac{-T}{R_s C_{sa}}\right) = P_{sa}\theta$$

$$\Delta P_{sa} = \frac{\Delta V_0}{C_{sa}} = P_{sa}(0) - P_{sa}(T)$$

- Two equations and two unknowns for $P_{sa}(0)$ and $P_{sa}(T)$.
- Note $0 < \theta < 1$, the solutions of idealized arterial pulse are,

$$P_{sa}(0) = \frac{\nabla V_0}{C_{sa}(1-\theta)}$$

$$P_{sa}(T) = \frac{\nabla V_0 \theta}{C_{sa}(1-\theta)}$$

given $\theta = \exp\left(\frac{-T}{R_s C_{sa}}\right)$.

1.12 Computer Simulation of Pulsatile Blood Flow

- Solve the equations of time dependent pressure, flow, and volume in the circulatory system.
- The systemic arterial pulse
- Governing equations,

$$\dot{V} = \frac{dV}{dt} = Q_1 - Q_2$$

$$\dot{V}_{sa} = Q_{A_0} - Q_s \qquad (1)$$

$$V_{sa} = V_{sa_d} + C_{sa}P_{sa} \qquad (2)$$

$$Q_s = {P_{sa}}/{R_s} \qquad (3)$$

- 3 unknowns $V_{sa}(t)$: *the systemic arterial volume*
  $P_{sa}(t)$: *systemic arterial pressure*
  $Q_s(t)$: *flow via systemic resistance (outflow)*
- Given known constants Q_{A_0}: inflow to the systemic arteries
  V_{sa_d}: systemic artery volume at $P_{sa} = 0$
  C_{sa} : compliance of systemic arteries
  R_s: systemic resistance
- $\dot{V}_{sa} = Q_{A_0} - Q_s$: conservation of volume for the systemic arteries. The rate of volume is the inflow minus the outflow.
- $V_{sa} = V_{sa_d} + C_{sa}P_{sa}$: compliance relationship between volume and pressure
- $Q_s = {P_{sa}}/{R_s}$: flow through the systemic resistance
- By substituting (2) and (3) into (1), and drop V_{sa_d} (=0)

$$C_{sa}\dot{P}_{sa} = Q_{A_0} - {P_{sa}}/{R_s}$$

- *Important*

  $Q_{A_0} = Q_{A_0}(t)$ given in Fig. 1.14

  $Q_{A_0}(t) = Q_{A_0}(t+T)$ a periodic function of time where T is a period of the heartbeat and 1/T is the heart rate.

- Express then Fig. 1.14 as a set of equations

$$Q_{A_0}(t) = \begin{cases} \frac{Q_{max}t}{T_{max}}, & 0 \le t \le T_{max} \\ \frac{Q_{max}(T_s - t)}{T_s - T_{max}}, & T_{max} \le t \le T_s \\ 0, & T_s \le t \le T \end{cases} \quad \text{(see Fig. 1.14)}$$

where Q_{max}: the maximum flow

T_{max}: the time of which maximum flow occurs

T_s : the duration of systole

stroke volume = integral of Q_{A_0} over one cycle = $Q_{max}T_s/2$

Cardiac output = $Q_{max}T_s/2T$

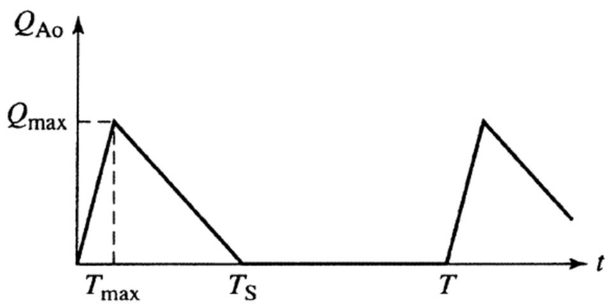

Figure 1.14. Outflow from the left ventricle trough the aortic valve into the systemic arterial tree is here denoted $Q_{Ao}(t)$. It is a periodic function whose triangular waveform is determined by the parameters Q_{max} = peak flow, T_{max} = time of peak flow, T_S = duration of systole, and T = duration of heartbeat. Note that the stroke volume is given by the integral of $Q_{Ao}(t)$ over one heartbeat, which is equal to $Q_{max}T_S/2$, and that the cardiac output is the stroke volume times the heart rate, which is therefore equal to $Q_{max}T_S/(2T)$.

- For Q_{A_0}, T=0.0125min, T_s=0.0050min, T_{max} = 0.002 min, Q_{max} = 28L/min, then stroke volume = 0.07L and cardiac output = 5.6L/min

Numerical Solution via Backward Euler Method

- Numerical Solution of $C_{sa}\dot{P}_{sa} = Q_{A_0} - {}^{P_{sa}}/_{R_s}$

  Replace the time derivative by a difference quotient.

$$\frac{C_{sa}[P_{sa}(t) - P_{sa}(t-\Delta t)]}{\Delta \mathrm{t}} = Q_{A_0}(t) - {}^{P_{sa}(t)}/_{R_s}$$

$$P_{sa}(t) = \frac{P_{sa}(t-\Delta t) + \Delta t Q_{A_0}(t)/C_{sa}}{1 + \Delta t/(R_s C_{sa})}$$

  given $P_{sa}(0)$, compute $P_{sa}(\Delta t)$, $P_{sa}(2\Delta t)$, …

- Questions
  1) $P_{sa}(0)$ and Δt, how to choose?
  2) $P_{sa}(0)$ any value for transition to steady state?
  3) Δt=? Small enough? what happens to the approximation $\frac{[P_{sa}(t)-P_{sa}(t-\Delta t)]}{\Delta \mathrm{t}}$=$\dot{P}_{sa}$