Pattern Classification Homework #4

Two normal distribution are characterized by

$p(\underline{x}|S_i)=N(\underline{x},\underline{m}_i,\Sigma_i)$, i=1,2

$P(S_1)=P(S_2)=0.5$

$m_1=[1,0]^T$

$m_2=[-1,0]^T$

$\Sigma_1=\Sigma_2=\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$

(a) Draw the Bayes decision boundary which minimized the probability of error.

(b) Draw the Bayes decision boundary which minimized risk when

$C_{11}= C_{22}= 0$ and $C_{12}=2C_{21}$

(c) Repeat (a) and (b) for

$\Sigma_1=\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$, $\Sigma_2=\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$

(a) Bayes decision rule for minimizing the probability error

$P(S_1|x) > P(S_2|x)\ x\in S_1$

$P(S_1|x) < P(S_2|x)\ x\in S_2$

$$\frac{P(x|S_1)}{P(x|S_2)} \begin{matrix} > \\ < \end{matrix} \frac{P(S_2)}{P(S_1)}$$

Since $P(S_2) = P(S_1)=0.5$

$$\frac{P(x|S_1)}{P(x|S_2)} \begin{matrix} > \\ < \end{matrix} 1$$

Take log on both sides

$$[\ln P(x|S_1) - \ln P(x|S_2)] \begin{matrix} > \\ < \end{matrix} 0$$

$g = \ln P(x|S_1) - \ln P(x|S_2) = -1/2\ (x-m_1)^T\Sigma_1^{-1}(x-m_1) + -1/2\ (x-m_2)^T\Sigma_2^{-1}(x-m_2)$

$= -1/2[x_1-1,x_2]\ 4/3 \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} x_1-1 \\ x_2 \end{bmatrix}+1/2[x_1+1,x_2]\ 4/3 \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} x_1+1 \\ x_2 \end{bmatrix}$

$= (8/3)\ x_1 - (4/3)\ x_2$

Therefore $(8/3)\ x_1 - (4/3)\ x_2 \gtrless 0$ or $x_1 \gtrless 1/2\ x_2$ or $x_2 \gtrless 2\ x_1$

$g=x_2-2x_1=0$, $x_2=2x_1$ is decision boundary

(b) Bayes decision boundary for minimizing risk

$C_{11}=C_{22}=0$ and $C_{12}=2C_{21}$

Decision rule

$$P(x|S_1) / P(x|S_2) \gtrless (C_{12}-C_{22})P(S_2) / (C_{21}-C_{11})P(S_1)$$

$$P(x|S_1) / P(x|S_2) \gtrless (C_{12})P(S_2) / (C_{21})P(S_1) = 2C_{21}P(S_2) / C_{21}P(S_1) = 2$$

$$[\ \ln P(x|S_1) - \ln P(x|S_2)\ ] \gtrless \ln 2$$

$$8/3\ x_1 - 4/3\ x_2 \gtrless \ln 2 \Rightarrow [2x_1-x_2] \gtrless 3/4 \ln 2$$

Decision boundary $2x_1-x_2=3/4 \ln 2$

(c) $\Sigma_1^{-1}=4/3\begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}, \Sigma_2^{-1}=4/3\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$

Repeat the same as in (a)

$$g = \ln P(x|S_1) - \ln P(x|S_2) = -1/2\ (x-m_1)^T\Sigma_1^{-1}(x-m_1) + -1/2\ (x-m_2)^T\Sigma_2^{-1}(x-m_2)$$

$= 8/3\ x_1 + 4/3\ x_1x_2$

Therefore $8/3\ x_1 + 4/3\ x_1x_2 \gtrless 0$

Decision boundary $x_1(8/3 + 4/3\ x_2) = 0 \Rightarrow x_1=0$ or $x_2=-2$

From (b) and (c)

$$8/3\ x_1 + 4/3\ x_1x_2 \underset{<}{\overset{>}{}} \ln 2$$

Therefore $2x_1 + x_1x_2 \underset{<}{\overset{>}{}} 3/4 \ln 2$

Decision boundary

$2x_1 + x_1x_2 = 3/4 \ln 2$

$x_1(2+x_2) = 3/4 \ln 2$

$2+x_2 = (3/4)(\ln 2)(1/x_1)$

$x_2=(3/4)(\ln 2)(1/x_1) - 2$