IMLPR Homework 08 Solution

Two normal distribution are characterized by

$p(\underline{x}|S_i)=N(\underline{x},\underline{m}_i,\Sigma_i)$, i=1,2

$P(S_1)=P(S_2)=0.5$

$m_1=[1,0]^T$

$m_2=[-1,0]^T$

$\Sigma_1=\Sigma_2=\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$

(a) Find the Bayes decision boundary which minimized the probability of error.

(b) Repeat (a) for

$\Sigma_1=\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$, $\Sigma_2=\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$

(a) Bayes decision rule for minimizing the probability error

$P(S_1|x) > P(S_2|x)\ x \in S_1$

$P(S_1|x) < P(S_2|x)\ x \in S_2$

$$\frac{P(x|S_1)}{P(x|S_2)} \gtrless \frac{P(S_2)}{P(S_1)}$$

Since $P(S_2) = P(S_1)=0.5$

$$\frac{P(x|S_1)}{P(x|S_2)} \gtrless 1$$

Take log on both sides

$$[\ln P(x|S_1) - \ln P(x|S_2)] \gtrless 0$$

$g = \ln P(x|S_1) - \ln P(x|S_2) = -1/2\ (x-m_1)^T\Sigma_1^{-1}(x-m_1) + -1/2\ (x-m_2)^T\Sigma_2^{-1}(x-m_2)$

$= -1/2[x_1-1,x_2]\ 4/3 \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} x_1-1 \\ x_2 \end{bmatrix}+1/2[x_1+1,x_2]\ 4/3 \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} x_1+1 \\ x_2 \end{bmatrix}$

$= (8/3)\ x_1 - (4/3)\ x_2$

Therefore $(8/3)\, x_1 - (4/3)\, x_2 \underset{<}{>} 0$ or $x_1 \underset{<}{>} 1/2\, x_2$ or $x_2 \underset{<}{>} 2\, x_1$

$g = x_2 - 2x_1 = 0$, $x_2 = 2x_1$ is decision boundary

(b)

$\Sigma_1^{-1} = 4/3 \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}$, $\quad \Sigma_2^{-1} = 4/3 \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$

Repeat the same as in (a)

$g = \ln P(x|S_1) - \ln P(x|S_2) = -1/2\, (x-m_1)^T \Sigma_1^{-1} (x-m_1) + -1/2\, (x-m_2)^T \Sigma_2^{-1} (x-m_2)$

$= 8/3\, x_1 + 4/3\, x_1 x_2$

Therefore $8/3\, x_1 + 4/3\, x_1 x_2 \underset{<}{>} 0$

Decision boundary $x_1(8/3 + 4/3\, x_2) = 0 \Rightarrow x_1 = 0$ or $x_2 = -2$