the real world in a way that is admittedly not exact, but that leads to useful solutions to real problems.

From the above discussion, it appears that the most useful approach to probability for engineers is a two-pronged one, in which the relative-frequency concept is employed to relate simple results to physical reality, and the axiomatic approach is employed to develop the appropriate mathematics for more complicated situations. It is this philosophy that is presented here.

**1–4 The Relative-Frequency Approach**

As its name implies, the relative-frequency approach to probability is closely linked to the frequency of occurrence of the defined events. For any given event, the frequency of occurrence is used to define a number called the *probability* of that event and this number is a measure of how likely that event is. Usually, these numbers are assumed, the assumed values being based on our intuition about the experiment or on the assumption that the events are equally likely.

To make this concept more precise, consider an experiment that is performed \( N \) times and for which there are four possible outcomes that are considered to be the elementary events \( A, B, C, \) and \( D. \) Let \( N_A \) be the number of times that event \( A \) occurs, with a similar notation for the other events. It is clear that

\[
N_A + N_B + N_C + N_D = N
\]  

(1–1)

We now define the relative frequency of \( A, r(A) \) as

\[
r(A) = \frac{N_A}{N}
\]

(1–2)

From (1–1) it is apparent that

\[
r(A) + r(B) + r(C) + r(D) = 1
\]  

(1–3)

Now imagine that \( N \) increases without limit. When a phenomenon known as *statistical regularity* applies, the relative frequency \( r(A) \) tends to stabilize and approach a number, \( \Pr(A) \), that can be taken as the probability of the elementary event \( A. \) That is

\[
\Pr(A) = \lim_{N \to \infty} r(A)
\]

(1–3)

From the relation given above, it follows that

\[
\Pr(A) + \Pr(B) + \Pr(C) + \cdots + \Pr(M) = 1
\]  

(1–4)

and we can conclude that the sum of the probabilities of all of the mutually exclusive events associated with a given experiment must be unity.
These concepts can be summarized by the following set of statements:

1. $0 \leq \Pr (A) \leq 1$.
2. $\Pr (A) + \Pr (B) + \Pr (C) + \cdots + \Pr (M) = 1$, for a complete set of mutually exclusive events.
3. An impossible event is represented by $\Pr (A) = 0$.
4. A certain event is represented by $\Pr (A) = 1$.

To make some of these ideas more specific, consider the following hypothetical example. Assume that a large bin contains an assortment of resistors of different sizes, which are thoroughly mixed. In particular, let there be 100 resistors having a marked value of $1 \, \Omega$, 500 resistors marked $10 \, \Omega$, 150 resistors marked $100 \, \Omega$, and 250 resistors marked $1000 \, \Omega$. Someone reaches into the bin and pulls out one resistor at random. There are now four possible outcomes corresponding to the value of the particular resistor selected. To determine the probability of each of these events we assume that the probability of each event is proportional to the number of resistors in the bin corresponding to that event. Since there are 1000 resistors in the bin all together, the resulting probabilities are

$$
\begin{align*}
\Pr (1 \, \Omega) &= \frac{100}{1000} = 0.1 \\
\Pr (10 \, \Omega) &= \frac{500}{1000} = 0.5 \\
\Pr (100 \, \Omega) &= \frac{150}{1000} = 0.15 \\
\Pr (1000 \, \Omega) &= \frac{250}{1000} = 0.25
\end{align*}
$$

Note that these probabilities are all positive, less than 1, and do add up to 1.

Many times one is interested in more than one event at a time. If a coin is tossed twice, one may wish to determine the probability that a head will occur on both tosses. Such a probability is referred to as a joint probability. In this particular case, one assumes that all four possible outcomes ($HH$, $HT$, $TH$, and $TT$) are equally likely and, hence, the probability of each is one-fourth. In a more general case the situation is not this simple, so it is necessary to look at a more complicated situation in order to deduce the true nature of joint probability. The notation employed is $\Pr (A, B)$ and signifies the probability of the joint occurrence of events $A$ and $B$.

Consider again the bin of resistors and specify that in addition to having different resistance values, they also have different power ratings. Let the different power ratings be $1 \, \text{W}$, $2 \, \text{W}$, and $5 \, \text{W}$; the number having each rating is indicated in Table 1–2.

Before using this example to illustrate joint probabilities, consider the probability (now referred to as a marginal probability) of selecting a resistor having a given power rating without regard to its resistance value. From the totals given in the right-hand column, it is clear that these probabilities are

$$
\begin{align*}
\Pr (1 \, \text{W}) &= \frac{440}{1000} = 0.44 \\
\Pr (2 \, \text{W}) &= \frac{200}{1000} = 0.20 \\
\Pr (5 \, \text{W}) &= \frac{360}{1000} = 0.36
\end{align*}
$$
We now ask what the joint probability is of selecting a resistor of 10 Ω having 5-W power rating. Since there are 150 such resistors in the bin, this joint probability is clearly

\[ \Pr(10 \Omega, 5 \text{ W}) = \frac{150}{1000} = 0.15 \]

The 11 other joint probabilities can be determined in a similar way. Note that some of the joint probabilities are zero (for example, \( \Pr(1 \Omega, 5 \text{ W}) = 0 \)) simply because a particular combination of resistance and power does not exist.

It is necessary at this point to relate the joint probabilities to the marginal probabilities. In the example of tossing a coin two times, the relationship is simply a product. That is,

\[ \Pr(H, H) = \Pr(H) \times \Pr(H) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \]

But this relationship is obviously not true for the resistor bin example. Note that

\[ \Pr(5 \text{ W}) = \frac{360}{1000} = 0.36 \]

and it was previously shown that

\[ \Pr(10 \Omega) = 0.5 \]

Thus,

\[ \Pr(10 \Omega) \times \Pr(5 \text{ W}) = 0.5 \times 0.36 = 0.18 \neq \Pr(10 \Omega, 5 \text{ W}) = 0.15 \]

and the joint probability is not the product of the marginal probabilities.

To clarify this point, it is necessary to introduce the concept of conditional probability. This is the probability of one event \( A \), given that another event \( B \) has occurred; it is designated as \( \Pr(A|B) \). In terms of the resistor bin, consider the conditional probability of selecting a 10-Ω resistor when it is already known that the chosen resistor is 5 W. Since there are 360 5-W resistors, and 150 of these are 10 Ω, the required conditional probability is

\begin{table}
\centering
\caption{Resistance Values and Power Ratings}
\begin{tabular}{|c|rrrrr|}
\hline
Power Rating & 1 Ω & 10 Ω & 100 Ω & 1000 Ω & Totals \\
\hline
1 W & 50 & 300 & 90 & 0 & 440 \\
2 W & 50 & 50 & 0 & 100 & 200 \\
5 W & 0 & 150 & 60 & 150 & 360 \\
Totals & 100 & 500 & 150 & 250 & 1000 \\
\hline
\end{tabular}
\end{table}
Now consider the product of this conditional probability and the marginal probability of selecting a 5-W resistor.

\[ \Pr (10 \Omega | 5 \text{W}) \Pr (5 \text{W}) = 0.417 \times 0.36 = 0.15 = \Pr (10 \Omega, 5 \text{W}) \]

It is seen that this product is indeed the joint probability.

The same result can also be obtained another way. Consider the conditional probability

\[ \Pr (5 \text{W} | 10 \Omega) = \frac{150}{500} = 0.30 \]

since there are 150 5-W resistors out of the 500 10-Ω resistors. Then form the product

\[ \Pr (5 \text{W} | 10 \Omega) \Pr (10 \Omega) = 0.30 \times 0.5 = \Pr (10 \Omega, 5 \text{W}) \quad (1-5) \]

Again, the product is the joint probability.

The foregoing ideas concerning joint probability can be summarized in the general equation

\[ \Pr (A, B) = \Pr (A | B) \Pr (B) = \Pr (B | A) \Pr (A) \quad (1-6) \]

which indicates that the joint probability of two events can always be expressed as the product of the marginal probability of one event and the conditional probability of the other event given the first event.

We now return to the coin-tossing problem, in which it is indicated that the joint probability can be obtained as the product of two marginal probabilities. Under what conditions will this be true? From equation (1–6) it appears that this can be true if

\[ \Pr (A | B) = \Pr (A) \quad \text{and} \quad \Pr (B | A) = \Pr (B) \]

These statements imply that the probability of event A does not depend upon whether or not event B has occurred. This is certainly true in coin tossing, since the outcome of the second toss cannot be influenced in any way by the outcome of the first toss. Such events are said to be statistically independent. More precisely, two random events are statistically independent if and only if

\[ \Pr (A, B) = \Pr (A) \Pr (B) \quad (1-7) \]

The preceding paragraphs provide a very brief discussion of many of the basic concepts of discrete probability. They have been presented in a heuristic fashion without any attempt to justify them mathematically. Instead, all of the probabilities have been formulated by invoking the concepts of relative frequency and equally likely events in terms of specific numerical
examples. It is clear from these examples that it is not difficult to assign reasonable numbers to the probabilities of various events (by employing the relative-frequency approach) when the physical situation is not very involved. It should also be apparent, however, that such an approach might become unmanageable when there are many possible outcomes to any experiment and many different ways of defining events. This is particularly true when one attempts to extend the results for the discrete case to the continuous case. It becomes necessary, therefore, to reconsider all of the above ideas in a more precise manner and to introduce a measure of mathematical rigor that provides a more solid footing for subsequent extensions.

Exercise 1–4.1

a) A box contains 50 diodes of which 10 are known to be bad. A diode is selected at random. What is the probability that it is bad?

b) If the first diode drawn from the box was good, what is the probability that a second diode drawn will be good?

c) If two diodes are drawn from the box what is the probability that they are both good?

Answers: 39/49, 156/245, 1/5

(Note: In the exercise above, and in others throughout the book, answers are not necessarily given in the same order as the questions.)

Exercise 1–4.2

A telephone switching center survey indicates that one of four calls is a business call, that one-tenth of business calls are long distance, and one-twentieth of nonbusiness calls are long distance.

a) What is the probability that the next call will be a nonbusiness long-distance call?

b) What is the probability that the next call will be a business call given that it is a long-distance call?

c) What is the probability that the next call will be a nonbusiness call given that the previous call was long distance?

Answers 3/80, 3/4, 2/5