# Consumer Referral in a Small World Network

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**Abstract**

We consider a monopolist's profit-maximization problem when the demand for the product is created by the referrals between consumers who are connected over the social network. The monopolist decides the price of the product and the referral fee which is awarded to consumers whose referrals lead to sales of the product. It is known that the social network in the real world has the small world property that the average distance between any two players in the network is very short. We generate a small world network by applying the rewiring algorithm of Watts and Strogatz and incorporate it into the monopolist's profit maximization problem. Main results of the paper are (i) that contrary to our anticipation, the small world property of the underlying social network is not inherited to the monopolist's profit, in other words, the monopolist's profit smoothly (not rapidly) approaches the profit in the complete network as the rewiring probability increases, i.e., the network becomes denser, (ii) that the per consumer profit inherits the small world property, so that it is saturated to a finite non-zero value, when the rewiring probability is near zero, and (iii) that as a network becomes denser, the optimal referral fee tends to increase when the rewiring probability is small and then decrease for the rest of the rewiring probability values, while the optimal price remains stable above a threshold level of the probability. Overall, optimal values for the price and the referral fee are more significantly affected by the network density, but not by the network size very much.

*Key words: Consumer referral, Referral fee, Small world network, Network marketing*

*JEL Classification: D40, L10*

# 1 Introduction

Agents in a society interact (or communicate) with each other, but the pattern of interactions is not uniform across pairs of agents. Some pairs interact more often than others, because they are closer in terms of either geographical distance or social distance.[1] Then, a person $A$, for example, who is starting a business dealing with another $B$ may find it to his advantage utilizing his connection with the third person $C$ who is closer to $B$ in the sense that he has more casual interactions with $B$.

The pattern of interactions can be represented by a network indicating how each agent is linked directly to some agents and indirectly to others. It is known that the social network has the characteristic of so-called "small world phenomenon," namely that any two people can establish contact by going through only a short chain of intermediate acquaintances.[2] Recently, this property of social networks was modeled as a small-world network by Watts and Strogatz (1998).

Since Watts and Strogatz, there has been a large body of literature characterizing the small world network, but studies exploring its implications in the resultant social behavior of agents are rare. This article examines the performance of network marketing relying entirely on referrals by consumers in a small world network.[3]

By *network marketing* – often called "multi-level marketing" or "referral marketing" as well – we mean that a firm sells its products initially only to a certain number of consumers who are directly linked to the firm, and subsequently continues to sell to consumers who are referred by previous consumers and so on. In network marketing, the sales volume of a

[1] The term of social distance is defined by the Encyclopedea of Psychology (2000) as the perceived distance between individuals or groups. The concept has a long history in social science. See, for example, Borgardus (1928).

[2] Following the work of Milgram (1967), it is widely touted that the average number of acquaintances is about six, in other words, a person can reach any other in the world via only six intermediate acquaintances.

[3] See Jun and Kim (2002) for an economic analysis of network marketing in a simple path network.

firm will depend on the probability that each consumer will refer his neighbor. This probability is, by nature, not exogenously given, but endogenously determined from a consumer's comparisons between the benefit and the cost of his making a referral. To increase the probability, the firm will have an incentive to pay a reasonably high referral fee. In other words, on one hand, a high referral fee directly reduces a firm's profit but, on the other hand, it indirectly increases the profit by inducing more consumers to buy. So, a shorter average distance between any two consumers will increase the firm's profit, since it can alleviate the total expenditures on referral fees. In particular, if the consumer population is characterized by the complete network, the profit will be maximal.

Main questions we pose here are two-fold; (i) how the network-marketing monopolist can achieve the maximal profit given the network structure and (ii) how divergent the outcome achieved in a small world network is from the outcome in a complete network. Any network-marketing seller must be aware of the fact that referrals between consumers are constrained by the structure of social networks. Therefore to answer the first question we pose, it is necessary to incorporate the social network into the profit maximization framework of the seller. To answer the second question, in addition, we must be able to construct a small world network which will be geared into our model of profit maximization.

The challenge of embedding profit maximization in the social network is that the real-world social network is too complicated for an analytic treatment. For this reason, the economic literature obscures fundamental sociological features of transactions. The nature of the complexity comes from the fact that the real-world social network has a distinguishing property that is not possessed by an easy-to-study completely ordered or random network. The real-world social network is characterized by the *small world* property that, on one hand, it has small average path length in comparison with a completely ordered network and, on the other hand, it has higher clustering than a random network. However, it is possible to get rough answers to our questions if we were equipped with an appropriate network model by which we can simulate networks that resemble social networks. For this purpose, we will adapt the algorithm developed by Watts and Strogatz (1998) or Watts (1999). Watts and Strogatz (1998) were able to create a small world network by deleting and rewiring some of the links in a regular network, thereby transforming it into a small world network.

Specifically, the model is set up as follows. There is a monopolist and a finite number of potential consumers. In a social network formed among consumers, each consumer is represented by a node and the social acquaintance between consumers is represented by a link between nodes. The monopolist is only linked with one consumer who is informed of the product directly from the monopolist. The rest of the consumers are unaware of the product until they get referred by another consumer. Each consumer may refer consumers to whom he is linked, what we call "social neighbors". The monopolist can induce consumers to refer their neighbors by awarding them referral fees for successful referrals that lead to sales of the product. The profit-maximizing monopolist determines the price of the product and the referral fee at once.

The key simulation results of the paper are summarized as follows. First, we confirm that the rewiring algorithm indeed generates a small world network. More importantly, the per consumer profit of the monopolist in this small world network is saturated to a finite nonzero value as the population size is increased when the rewiring probability is near zero (but not exactly zero), although it decreases towards zero when the probability is zero. This implies that the small world property of underlying social network is inherited to the monopolist profit. Second, as the probability increases, the monopolist's profit smoothly approaches its profit under the complete network where all consumers are linked to one another. It is contrary to our conjecture that the monopolist's profit will approach its complete network profit very rapidly because even a fairly small rewiring probability can make the average distance between two consumers over the rewiring network very short. This simulation result suggests that the monopolist's profit decreases exponentially to the average distance, not linearly. Third, the small world transition is also reflected in the optimal pair of price and referral fee. The optimal price remains stable around the optimal level under the complete network as long as the rewiring probability is above a certain level which is near zero, The optimal referral fee tends to decrease above the threshold rewiring probability as the network converges to the complete network, while it tends to increase below it. Thus, this is also an evidence for the fact that the small world transition has occurred. Finally, the rank of the powerseller – who refers the largest number of neighbors successfully – is not monotone with respect to the rewiring probability.

The organization of the paper goes as follows. In Section 2, we describe a model of consumer referral in a deterministic network and summarize the results of Jun and Kim (2002). In Section 3, we introduce our model of a small world network. Simulation results are provided in Section 4.

## 2 A *Primer*

As a benchmark, we describe a simple model of consumer referral which was developed by Jun and Kim (2002). The model goes as follows.

Suppose there is a monopolist and a set of consumers $N = \{1, 2, \cdots, n\}$. The network relation among them is represented by a graph $g$. The monopolist, which we will call player 0, has no link with consumers except one, say, consumer 1.

The monopolist produces its goods at a constant marginal cost $c(> 0)$. Consumer $i$'s valuation for this product is $v_i$ which is independently and identically distributed over $[\underline{v}, \overline{v}]$ according to the continuous density function $f(\cdot)$ and the corresponding distribution function $F(\cdot)$. The value of $v_i$ is unknown to other consumers than consumer $i$. Each consumer buys at most one unit of the product. A consumer who buys one may refer consumers who are adjacent to him. The referral cost is $\delta(>0)$.

The game proceeds as follows. First, player 0 ("monopolist") offers the unit price $p$ and the referral fee $r$ to player 1. The prices, $p$ and $r$, will prevail until the game ends. Second, consumer 1 decides whether or not to purchase one at $p$ and $r$. If he does not purchase, the game terminates. If he purhcases one, he in turn decides whether to refer consumers who are linked. He must bear the referral cost regardless of the referral outcome, but he is paid a referral fee only if he can convince a linked consumer to buy one. The consumer who is convinced to buy one by his neighbor may refer other consumers. This procedure is continued until no consumer is left to refer. Note that we exclude the possibility that a consumer who does not buy himself refer his neighbors.[4] In this section, we assume that the game structure, the network structure and previous moves of other players are common knowledge to all players including player 0. As usual, the equilibrium concept for this sequential game is the "Subgame Perfect Equilibrium".

We summarize the results of Jun and Kim (2002), focusing on two extreme cases of the network structure.

## Local Network

The network structure of $g$ is represented by $(g_{ij})_{1\leq i\leq n-1, j>i}$ where $g_{ij} = alpha$ denotes "$i$ and $j$ are linked with probability $\alpha$."

Suppose $g$ is a path graph such that $g_{ij} = 1$ if $j = i+1$, $g_{ij} = 0$ otherwise, for all $1 \leq i \leq n-1$. In this local graph, a consumer communicates only with his geographical neighbor.

[4]This can be justified by the presumption that his recommendation would not be as credible as the recommendation by one who buys himself. Moreover, in network marketing, it is usual that only consumers who buy themselves refer other potential consumers. In Jun and Kim (2002), we identified a condition for this buy-to-refer constraint to be profitable to the monopolist.

Given $p$ and $r$, consumer $i$ purchases one if $v_i \geq \tilde{v}_i$ (minimum required valuations), where $\tilde{v}_i \equiv p - \max\{r[1 - F(\tilde{v}_{i+1})] - \delta, 0\}$, $i = 1, \cdots, n-1$ and $\tilde{v}_n = p$. Thus, a sequence of propositions follow.

**Proposition 1** *Suppose that $\tilde{v}_k = p$ and $\tilde{v}_{k-1} < p$ for some $k$. Then, the minimum required valuation increases as the chain of purchases proceeds:*

$$\tilde{v}_1 < \tilde{v}_2 < \cdots < \tilde{v}_k = \cdots = \tilde{v}_n = p.$$

*In particular, if $k = n$,*

$$\tilde{v}_1 < \cdots < \tilde{v}_n = p.$$

*Proof.* See Jun and Kim (2002).

**Proposition 2** *[RC] $\Longleftrightarrow \tilde{v}_{n-1} < p \Longleftrightarrow \tilde{v}_1 < \cdots < \tilde{v}_{n-2} < p$, where [RC] is $r[1-F(p)] > \delta$.*

*Proof.* See Jun and Kim (2002).

Here, [RC] is the condition for which a buying consumer refers the next consumer. This proposition says that, if [RC] is satisfied, all consumers who buy one will always refer and, if not satisfied, no consumer will refer. Proposition 3 is a direct consequence of Proposition 2.

**Proposition 3** *$k = 1$ or $n$, where $k$ is the integer defined in Proposition 1.*

*Proof.* See Jun and Kim (2002).

This proposition says that only two outcomes are feasible in equilibrium: either consumer 1 does not refer ($k = 1$: *no referral equilibrium*), or if he refers, all the subsequent consumers who buy refer their neighbor ($k = n$: *all referral equilibrium*). Hence it cannot occur in equilibrium that a consumer, except consumer 1, does not refer his neighbor after he buys one himself.

From strategies of consumers characterized above, we can see that the monopolist's profit is either

$$\begin{aligned}\pi(n) &= (p-c)Q(n,1) + (p-c-r)\sum_{k=2}^{n} Q(n,k) \\ &= \sum_{k=1}^{n}\{k(p-c)-(k-1)r\}Q(n,k)F(\tilde{v}_{k+1})\end{aligned}$$

if $p$ and $r$ satisfy [RC] condition, or $(p-c)[1-F(p)]$ otherwise, where $Q(n,k) = \prod_{i=1}^{k}[1 - F(\tilde{v}_i)]$.

### Complete Network

A complete network can be represented by a graph $g$ such that $g_{ij} = 1$ for all $i < j$.

Since consumer $i$ will buy one if and only if $v_i \geq p$ for all $i > 1$, consumer 1 refers if and only if $r[1-F(p)] > \delta$ and refers all neighbors if he does refer any. Thus, consumer 1 will buy one if

$$v_1 \geq \tilde{v}_1 \equiv p - (n-1)\max\{r[1-F(p)]-\delta, 0\}.$$

Then, the problem of the monopolist is

$$\max_{p,r} \tilde{\pi} = (1-F(\tilde{v}_1))[(p-c)+(n-1)(p-c-r)\{1-F(p)\}]. \tag{1}$$

# 3 Incorporating Random Links

Suppose random links are added to the local network with or without deleting one existing link in such a way that $g_{ij} = \alpha \in (0,1)$ for all $i \geq 1$ and $j \geq i+2$. A plausible interpretation of a network with random components is that although consumers are fully aware of the network structure, the monopolist is not fully informed of the structure, but only knows that consumer $i$ and $j$ are linked with probability $\alpha$ for $j \geq i+2$.

## Illustration

There are two ways of adding random links, $\alpha$-algorithm whereby no link is deleted when a random link is added and $\beta$-algorithm (rewiring algorithm) whereby a link is deleted whenever a random link is added.

To illustrate $\alpha$-algorithm, consider the simplest case of $n = 4$. Since possible random links are 13, 14 and 24, there are following eight possible configurations.

$$
\begin{aligned}
g_0 &\equiv g \\
g_1 &\equiv g + \{13\} \\
g_2 &\equiv g + \{14\} \\
g_3 &\equiv g + \{24\} \\
g_4 &\equiv g + \{13, 14\} \\
g_5 &\equiv g + \{13, 24\} \\
g_6 &\equiv g + \{14, 24\} \\
g_7 &\equiv g + \{13, 14, 24\}
\end{aligned}
$$

Each probability of these configurations is $(1-\alpha)^3$, $\alpha(1-\alpha)^2$, $\alpha(1-\alpha)^2$, $\alpha(1-\alpha)^2$, $\alpha^2(1-\alpha)$, $\alpha^2(1-\alpha)$, $\alpha^2(1-\alpha)$, $\alpha^3$ in order. Also, each configuration generates a directed tree indicating the referral order. To avoid complication of multiple referrals to a consumer, we may assume that a consumer can get referred by at most one consumer, the earlier consumer. For example, in $g_5$, consumer 4 can get referred by consumer 2, not consumer 3. Due to this assumption, we can obtain referral trees corresponding to eight configurations. We call a tree corresponding to $g_i$ an induced tree from $g_i$.

On the other hand, in the case of rewiring algorithm, possible configurations of network include

$$
\begin{aligned}
g_0 &\equiv g \\
g_1 &\equiv g + \{13\} - \{23\} \\
g_2 &\equiv g + \{14\} - \{34\} \\
g_3 &\equiv g + \{24\} - \{34\} \\
g_4 &\equiv g + \{13\} + \{24\} - \{23\} - \{34\} \\
g_5 &\equiv g + \{13\} + \{14\} - \{23\} - \{34\}
\end{aligned}
$$

The advantage of constructing a network in this way is that it induces a referral tree without the assumption that the possibility of getting referred by multiple consumers is resolved for the earlier consumer.

It is known that the network resulting from either algorithm exhibit the small world property, even if the network generated by $\beta$-algorithm contains the much less number of links than the one by $\alpha$-algorithm. We will use the rewiring algorithm to construct our small world network simply because it is much easier to generate referral trees by the algorithm, although the interpretation seems less natural.[5]

## Definitions

Below, we give definitions that will be used for our analysis. A tree $T$ is a connected graph with no cycle. $N(T)$ is the set of nodes in tree $T$ and $ij \in T$ denotes the edge connecting nodes $i$ and $j$. If individual $i$ follows individual $j$, we say that $j$ *precedes* $i$. Let $P(i)$ denote the set of individuals who precede $i$. If $ij \in T$, we say that $i$ *immediately precedes* $j$. If $P(i) = \emptyset$, $i$ is called *primary individual*. Let $N_i \subset N \setminus \{i\}$ denote the set of individuals who $i$

[5]It is hard to interpret the requirement of $\beta$-algorithm that one link be deleted whenever a new link is added. One possible interpretation might be that each consumer has only a limited amount of resources for maintaining his links.

immediately precedes. We often call $N_i$ as the set of social "neighbors" to individual $i$.[6] By the definition of a tree, each individual has only one individual by whom he is immediately preceded (except the root of the tree). Individuals $i$ and $j$ can be also connected indirectly if there is a sequence of vertices $i, i_1, i_2, ..., i_k, j$ such that $ii_1, i_1i_2, ..., i_kj \in T$. Such a sequence is called a *path* between $i$ and $j$. The tree is *complete* if the individual is directly linked with the rest of the individuals, i.e., $N_{i_0} = N \setminus \{i_0\}$ where $i_0$ is the primary individual. We can easily see that the induced tree from the complete network is complete.

The *rank* of an individual $i$, denoted by $\gamma(i)$, is defined by the number of individuals who precedes $i$, that is, $\gamma(i) = \#P(i)$. The rank of the primary individual is 0. The rank of a tree $T$, $\gamma(T)$, is defined by the maximum rank, i.e., $\gamma(T) = \max\{\gamma(i) : i \in N(T)\}$. Note that the rank of a tree is similar to the diameter of a graph, and so it is useful to measure the maximum (or sometimes average) distance between individuals.

## Strategies of Consumers

A consumer's decision can be represented in a recursive manner as before: consumer $i$ buys if and only if

$$v_i \geq \tilde{v}_i \equiv p - \sum_{j \in N_i} \max\{r(1 - F(\tilde{v}_j)) - \delta, 0\} \text{ and } \tilde{v}_i = p \text{ if } N_i = \emptyset. \tag{2}$$

The monotonicity of the minimum valuations implied by Proposition 1 does not necessarily hold in a general network. For example, consider the network in Figure 1. Suppose $v_i$ is uniformly distributed over $[0, 1]$. Compare the minimum valuations of consumer 1 and 2: $\tilde{v}_1 = p - \max\{r(1 - \tilde{v}_2) - \delta, 0\}$ and $\tilde{v}_2 = p - \max\{r(1 - \tilde{v}_3) - \delta, 0\} - \max\{r(1 - \tilde{v}_4) - \delta, 0\}$ where $\tilde{v}_3 = \tilde{v}_4 = p - \max\{r(1 - p) - \delta, 0\}$. Thus, $\tilde{v}_1 > \tilde{v}_2$ if $r < \sqrt{2}/2$ and $r(1 - p) > \delta$.

In a general network, if the [RC] condition is met, all consumers who buy one will always refer. This is because $\tilde{v}_i$ must be lower than $p$ for all $i$. Hence to avoid complication, we will

[6] Note that the immediate predecessor of $i$ is not included in $i$'s social neighbors.

assume that [RC] condition is met.

### Strategies of the Monopolist

The expected profit of the monopolist is defined as the expected sum of profits over all possible networks. Let $\Phi(n)$ be the set of all possible tree networks with $n$ consumers. Given the consumers' behavior described above, the monopolist will choose the optimal price $p^*$ and the optimal referral fee $r^*$ (given that [RC] is met) to maximize the expected profits in the network-embedded transactions.

The monopolist's profit in a general tree network is

$$\pi_N(n) = \sum_{k \in \Phi(n)} \omega_k \pi_N^k(n) \tag{3}$$

where $\omega_k$ is the probability of the $k$-th possible network in $\Phi(n)$ and the profit of the monopolist in the network is

$$\pi_N^k(n, \delta) \equiv (p - c)(1 - F(\tilde{v}_1^k)) + (p - c - r) \sum_{i=2}^{n} \prod_{j \in P^k(i)} (1 - F(\tilde{v}_j^k))(1 - F(\tilde{v}_i^k)).$$

# 4 Simulations

## 4.1 Procedure

First, it is worthwhile to make clear why the analytic solution is not tractable in this model. The profit specified in equation (3) is based on computations of the contingent profit in all possible networks. When $n$ is large, the number of all possible networks is huge. In fact, it grows in a factorial of the network size.[7] Therefore it is not possible to enumerate all possible networks, which prevents a rigorous analytic treatment.[8]

[7] In fact, the problem belongs to the $\mathcal{NP}$-complete class, which is characterized by the extreme computational load.

[8] It is also unlikely that the monopolist will consider all possible networks in computing its expected profits.

The simulations proceed as follows. First, we generate a consumer network by applying the $\beta$-algorithm of Watts & Strogatz (1998) and Watts (1999). Fix the rewiring probability $\alpha$. Start with a line network $g_L$ where consumer 1 is directly connected to the monopolist, and $i(i+1) \in g_L$ for $i = 1, 2, ..., n-1$. Then we rewire $n-1$ links in $g_L$ as follows. Select consumer $i$, starting from 3 to $n$ chosen in turn, along with the edge that connects it to its immediate predecessor $i-1$. A random number $\omega$ between 0 to 1 is generated. If $\omega > \alpha$, do nothing and turn to the next consumer. If $\omega \leq \alpha$, then the edge between $i$ and $i-1$ is deleted and *rewired* to a consumer who is randomly chosen from the set $\{j | \gamma(j) < \gamma(i) - 1\}$ in the sequentially induced tree. The procedure of rewiring stops when all consumers have been considered once. Figure 2 illustrates the consumer network for various values of $\alpha$. When $\alpha = 0$, the initial line network is maintained, leading to a single-path tree. When $\alpha = 1$, all links are rewired, and the resulting network is the complete tree. Hence by tuning the rewiring probability, we can transform the line network into various forms of tree.

For our simulation, we assume that $F(.)$ is uniformly distributed on $[0, 1]$. If $p > 1$, [RC] condition cannot be satisfied. Also, if $r > p$, the monopolist will lose money by selling a unit. So, we confine the space of the price and referral fee to $[0, 1] \times [0, 1]$.

In the first round, we pick ten equally-spaced points from the two dimensional space $[0, 1]^2$, which sums up to 100 sample pairs of price and referral fee. We generate 10,000 random networks and compute the monopolist profit for each network by plugging each sample value of price and referral fee.[9] Then we compute the average price, referral fee, and profit of the monopolist.

In the second round, we repeat the procedure in the first round for 100 sample pairs of price and referral fee centered around the pair of price and referral fee which yielded the

[9] We also tried with 500,000 and 100,000 random networks, but the results are barely affected by the number of sample networks (as along as it is sufficiently large).

highest profit in the first round. We repeat this round 4 times.[10] Finally the pair of the price and referral fee yielding the highest profit is kept, and then we repeat the whole procedure for another value of $\alpha$. We do simulations for $n = 100, 200, 400$,and 800.

The simulations below are based on $c = 0.05$ and $\delta = 0.01$. For robustness, we experiment with different parameters, but the results seem to be robust to these variations.

## 4.2 Results

First, we check whether the constructed network exhibits the small world property. In the outer box of Figure 3, we depict the average rank of the induced tree versus the rewiring probability. It clearly shows that the average rank of the tree decreases from that of the single-path tree to that of the complete tree even when a few rewirings occurs. This shows that the fabricated social network by introducing random rewirings exhibits the small world property, as we predicted.

Given that the network exhibits the small world property, it is interesting to see how the profit changes as $\alpha$ increases. Our conjecture was that the profit $\pi(n)$ would increase very fast to the profit in the complete tree as $\alpha$ increases because the average rank is very short in a small world network. On the contrary to our anticipation, Figure 4 shows that the monopolist's profit approaches smoothly, not rapidly, to that in the complete tree. This suggests that the loss in the monopolist's profit increases not proportionally to the average rank, but exponentially. Intuitive explanation for this can be provided as follows. The profit in the complete tree is $\tilde{\pi} = (p-c)(1-F(\tilde{v}_1)) + (n-1)(p-c-r)(1-F(\tilde{v}_1))(1-F(p))$ as in equation (1). If the average distance from consumer 1 to any other consumer is $d$ in a small world network, the monopolist's profit can be approximated by $\pi_N \approx (p-c)(1-F(\tilde{v}'_1)) + (n-1)(p-c-r)\prod_{j\in P(i)\setminus\{1\}}(1-F(\tilde{v}_j))(1-F(p))$, where $\#P(i) \approx d-1$. Thus, the monopolist's profit decreases exponentially to the average rank, not linearly.

[10]Therefore the precision of the optimal price, referral fee and profit is $10^{-4}$.

However, the per consumer profit of the monopolist profit inherits the small world property from the underlying social network. Figure 5 shows that the per consumer profit decreases linearly as $n$ increases for $\alpha = 0$, but that it appears to be saturated to a finite non-zero value as soon as $P$ becomes non-zero. This shows that there is a small world transition when very few links are rewired.[11] If the network is complete, $\pi(n)$ increases linearly to $n$, so $\pi(n)/n$ is kept at some positive value. If the network exhibits the small world property resembling the complete network for $\alpha > 0$, $\pi(n)/n$ will still stay around a nonnegative value. If $\alpha = 0$, the network is not small world, so $\pi(n)$ grows less slowly than linearly as $n$ increases. Thus, $\pi(n)/n$ keeps decreasing linearly.

As $\alpha$ increases, the optimal referral fee tends to increase when $\alpha$ is small and then decrease for the rest of $\alpha$ values, while the optimal price remains stable above a threshold level of $\alpha$. The intuition is as follows. When the network is close to the line network, so most consumers have only one neighbor to refer, it is too costly for the monopolist to induce consumers later in the chain to get referred. Therefore in this case, the referee fee policy will not be an effective way of promotion. As an alternative measure, it will be more effective for the monopolist to charge a low price. When links are sufficiently rewired such that the average rank of network is close to that of complete network (for $\alpha \geq 0.1$), the referee fee policy becomes effective, so the monopolist will find it more profitable to lower the referral fee to induce more referrals with maintaining a high price. Thus, as $\alpha$ rises, the referral fee decreases while the price stays almost the same. The intuition behind this result is closely related to the property of the small world network. Once the average number of neighbors is high enough by a few rewiring, consumers can earn fairly high referral benefits given a unit referral fee. Therefore, as the rewiring probability increases more, the monopolist reduces the

[11] By the small world transition, we mean a change in the functional form of the average path length from $d \propto n$ to $d \propto \log n$ (or increasing much slower than in a logarithmic form). Although physicists mean by a small world network the latter regime including the complete network in a broad sense, we will use the term of "small world network" in a narrow sense by referring only to a situation where $\alpha$ is near zero.

referral fee without undermining the incentive to refer very much. Overall, the simulations on the optimal price and referral fee suggest that the monopolist should decrease the referral fee while maintaining the price of the product as the underlying network transits to a small world network.

Finally, it is intriguing to know who is the most successful referer. We call the consumer with the largest number of referrals that lead to a sale of the product as *powerseller*. Figure 9 shows the rank of the powerseller. It shows that before $\alpha$ reaches 0.08, the rank of the powerseller increases, and then decreases for $\alpha > 0.08$. The intuition for this observation is as follows. Basically, under this definition, the consumer with more neighbors is more likely to be the powerseller than those with less neighbors, if not always. An increase in $\alpha$ has two effects. First, it reduces the rank of the whole network. So, it is not surprising that the rank of the powerseller decreases for $\alpha > 0.08$. However, the more crucial factor determining the rank of the powerseller (than the rank of the network) is the average minimum valuation of his neighbors, in turn determined by the number of their neighbors. If $\alpha$ is zero, all consumers have only one neighbor, but as $\alpha$ increases from zero, consumers in the middle of the chain can have more neighbors which enables them to be the powerseller.

Overall, we can conclude that optimal values for the price and the referral fee in the monopolist's profit maximization problem in a small world network are significantly affected by the network density measured by $\alpha$, but not by the network size $n$ very much. The network size is, of course, an important determinant of the monopolist's profit. But the simulation results suggest that when it solves the profit maximization problem to determine the optimal prices, the monopolist should take the network density and various factors determining it more seriously; for example, how close consumers are to each other in a given community, how frequently they interact. Optimal decisions of the monopolist appears to be nearly independent of the network size.

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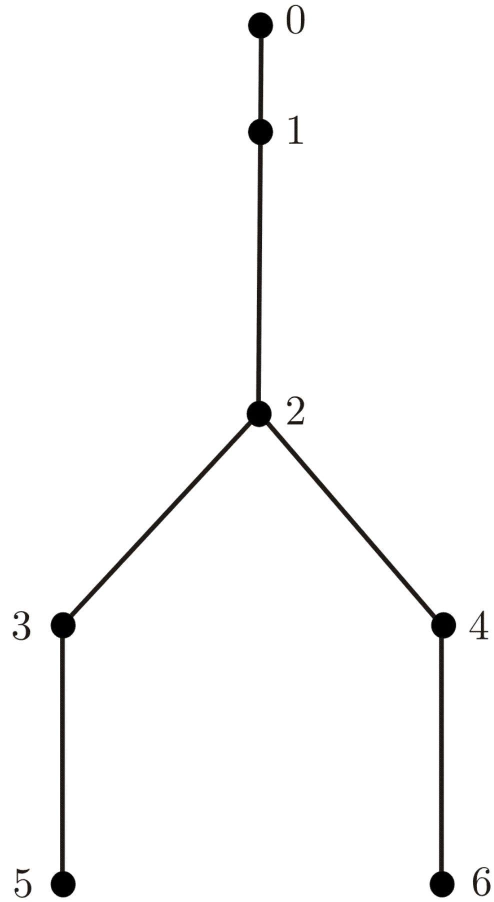
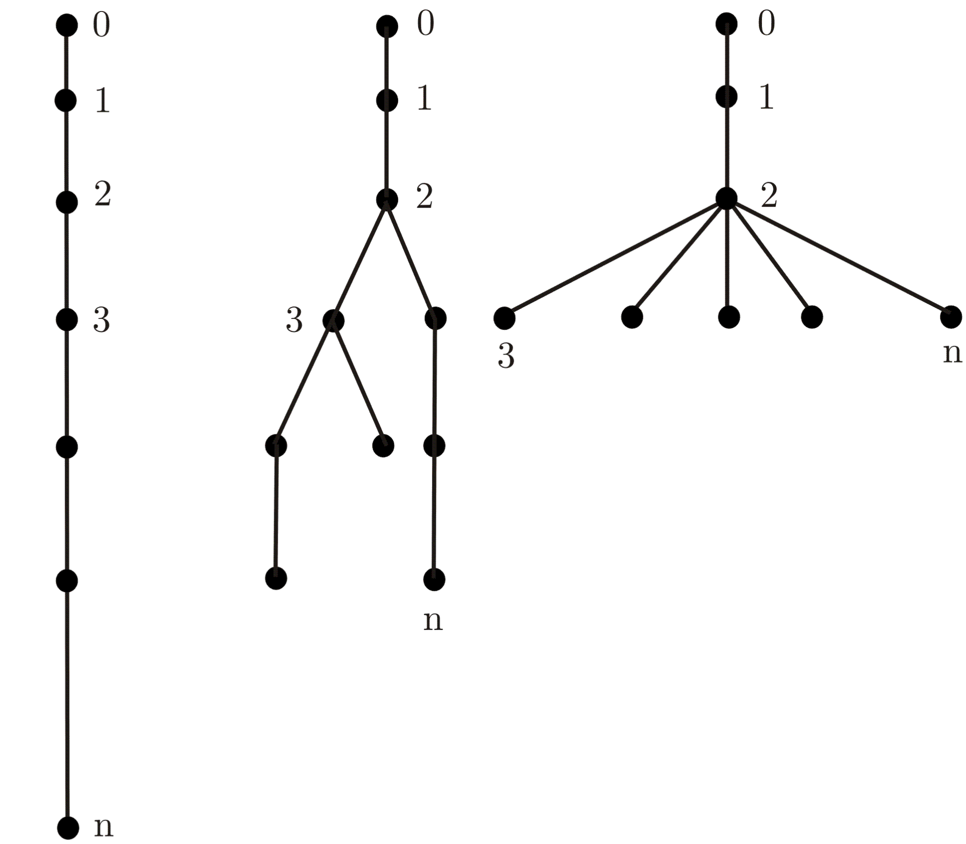
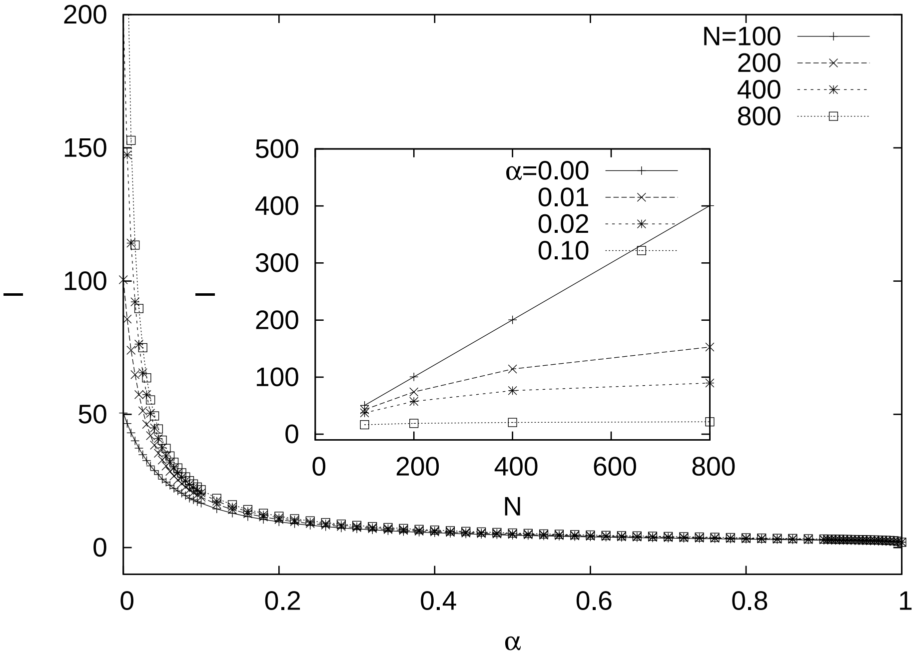



Figure 1: Non-monotonicity of the minimum valuation

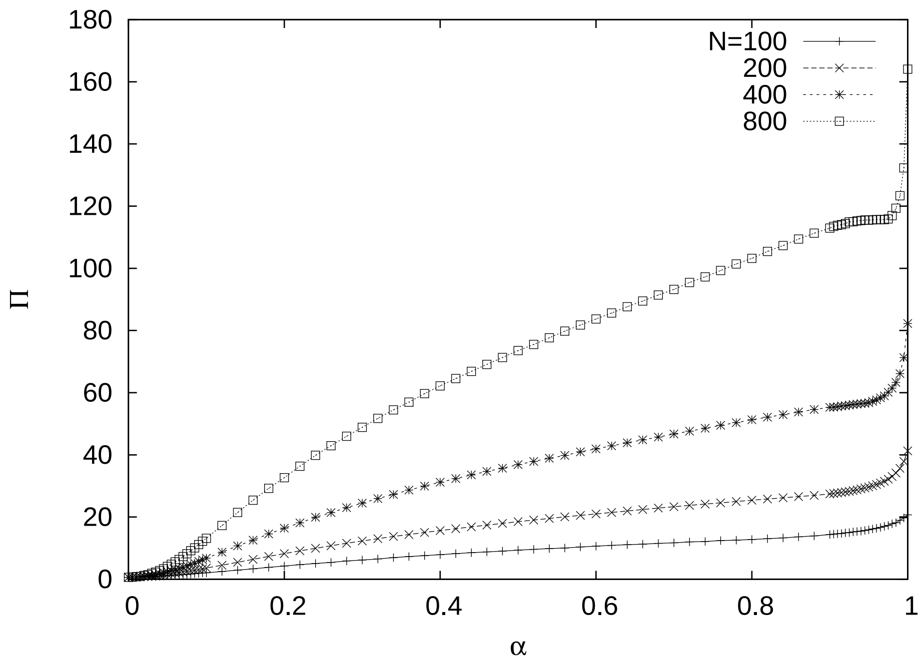



(a) P=0 (b) Intermediate P (c) P=1

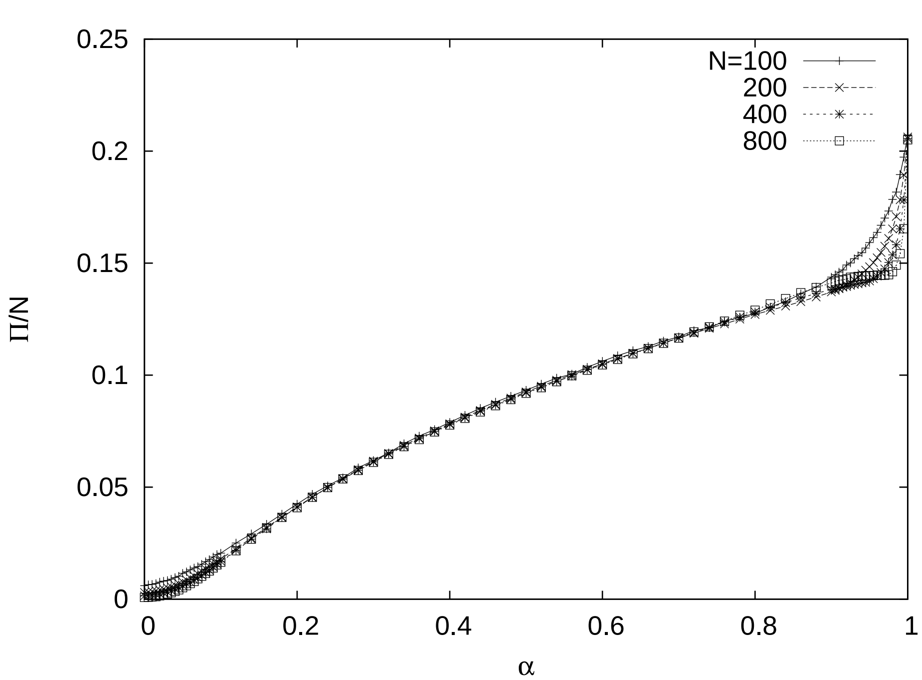
Figure 2: Various networks for different $P$



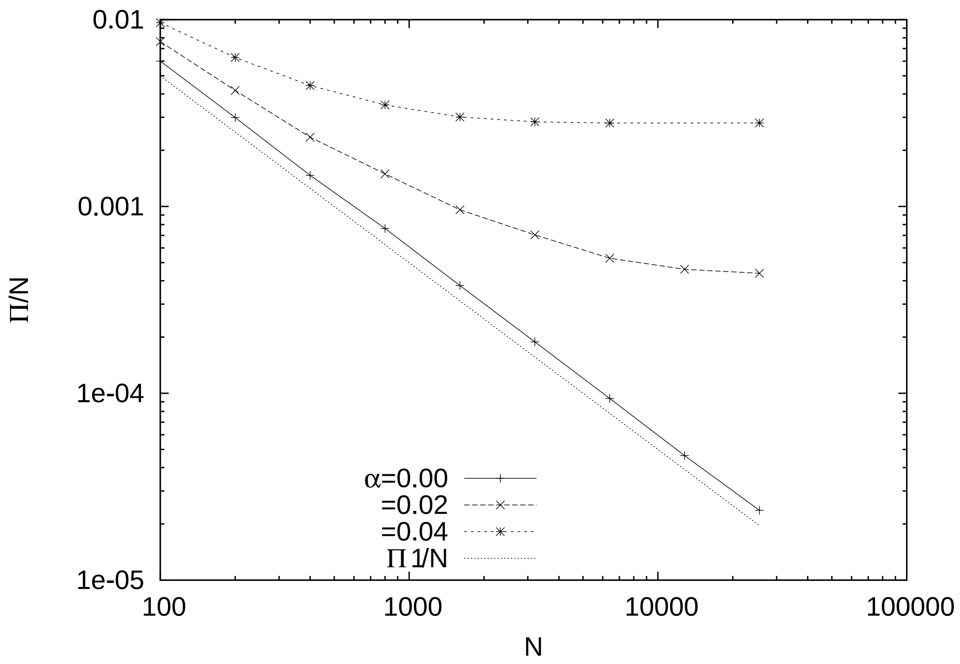
200
150
100
50
0
0
0.2
0.4
0.6
0.8
1
α
N=100
200
400
800
500
400
300
200
100
0
0
200
400
600
800
N
α=0.00
0.01
0.02
0.10



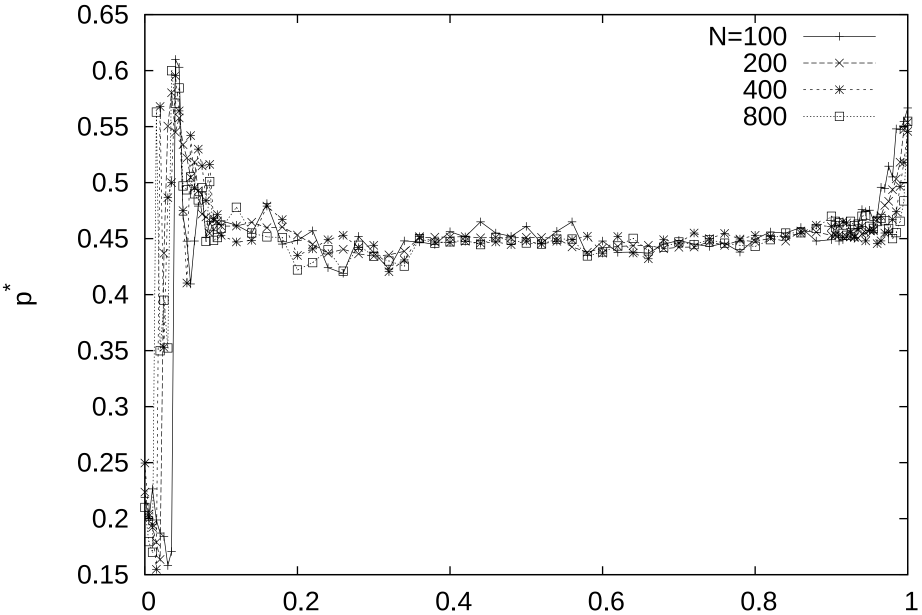
180
160
140
120
100
80
60
40
20
0
0
0.2
0.4
0.6
0.8
1
α
Π
N=100
200
400
800



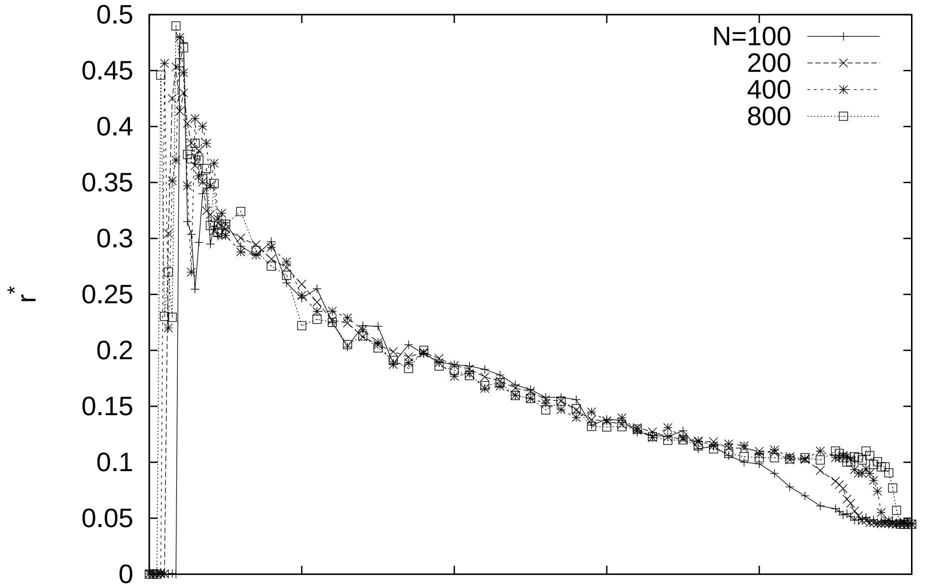
0.25
0.2
0.15
0.1
0.05
0
Π/N
0
0.2
0.4
0.6
0.8
1
α
N=100
200
400
800



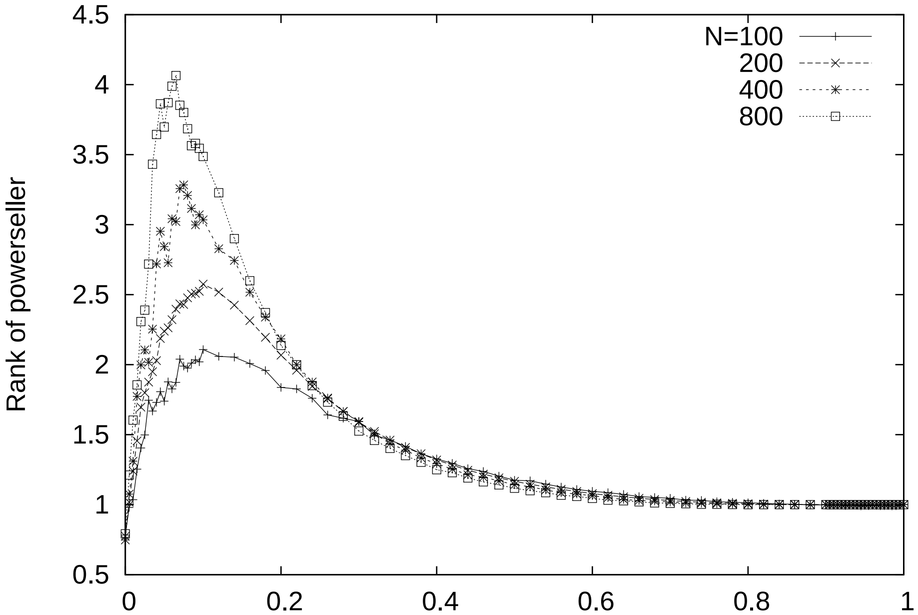
0.01
0.001
1e-04
1e-05
100
1000
10000
100000
N
Π/N
α=0.00
=0.02
=0.04
Π 1/N



N=100
200
400
800
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0
0.2
0.4
0.6
0.8
1
p*
α



0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
r*
0
0.2
0.4
0.6
0.8
1
α
N=100
200
400
800


4.5
4
3.5
3
2.5
2
1.5
1
0.5
Rank of powerseller
0
0.2
0.4
0.6
0.8
1
α
N=100
200
400
800