Joint Liability and Stochastic Shapley Value

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Abstract

If multiple defendants are jointly liable for a plaintiff's harm, the court must determine the apportionments of the damages among them. Recently, in a series of papers, Dehez and Ferey (2013, 2016a, 2016b) took a cooperative game theory approach, and used the Shapley value and the weighted Shapley value to determine the shares especially in the case of sequential acts. In this paper, we argue that their allocation rule is not fair if we take even a small random error into account. We alternatively propose the stochastic Shapley value which extends the definition of the Shapley value to stochastic cooperative games and show that it satisfies *ex post* efficiency, symmetry, dummy, feasibility, fairness and convergence to the Shapley value.

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Running Head: Joint Liability and Stochastic Shapley Value

1 Introduction

Harms are often caused by more than one injurer. Examples abound. Several factories emit toxical chemicals, causing water and ground pollution. Firms may engage in price-fixing col-

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lusion to extract consumers' surpluses. A person may be injured by a product manufactured unsafely by the contractor itself or its subcontractor. Traffic accidents commonly occur by multiple vehicle collisions.[1]

If multiple parties or factors are involved in an accident, it may be difficult to determine causation. A well known example is the case of Summers v. Tice that involves a plaintiff who was injured by one of two hunters shooting in their direction. In this case, it is simultaneity of the actions by two parties that makes it troublesome to determine causation, although one of them surely caused the accident. However, if there are other (random) factors that may have contributed to the accident, it becomes even more intricate to determine causation.

Once causation is established and so each of the defendants is found to be liable for the plaintiff's total harm, the defendants are called to be jointly and severally liable. Under the rule of joint and several liability, the plaintiff is allowed to recover the full amount of damages from any defendant, regardless of the particular injurer's share of the liability. The rule of joint and several liability is applied when either defendants acted jointly or the plaintiff's harm is indivisible. Depending on the apportionment rules, however, defendants may or may not take an action for contribution from the other defendants. Under the rule of no contribution, no one who paid more than their equitable share has the right to obtain any reimbursement from other defendants. On the other hand, under the rule of contribution, a defendant who paid more may obtain contribution from the other defendants. At common law, there was no right to contribution, but the contribution rule has been gradually replacing the no contribution rule from concerns that the no contribution rule may cause a serious consequence against the social justice. Recently, various statutes have explicitly provided for the contribution rule, and this trend creates a momentum for closely examining how to apportion the damage amount among several joint tortfeasors.

Most literature on joint tortfeasors has focused on providing a rationale for the joint and several liability rule from the aspect of (dynamic) efficiency,[2] i.e., whether it gives potential defendants a right incentive to take the efficient level of care.[3] This paper, aside from the efficiency issue, mainly addresses the issue of (static) fairness, what is a fair way

[1]For example, in Ybarra v. Spangard, the plaintiff was harmed during an operation under anesthesia jointly by doctors and nurses. Also, in the Agent Orange case, millions of people in Vietnam suffered illness due to the defoliant chemical made by several defendant companies.

[2]See, for example, Landes and Posner (1980), Shavell (1983), Kornhauser and Revesz (1989), Spier (1994), Yi (1991) and Kahan (1996).

[3]There is also vast literature that examines the efficiency of joint and several liability in terms of its effect on settlements. To name a few, see Kornhauser and Revesz (1994), Feess and Muehlheusser (2000) and Kim and Song (2007).

to apportion damages, given that care levels are already taken and the resulting damage amount is determined. The literature on this line is relatively scant. To the best of our knowledge, Ben-Shahar (1996, 2000) was the first to consider this issue by introducing the cooperative game theory approach. Later, Braham and van Hees (2009) used the concept of power indices. Recently, in a series of papers, Dehez and Ferey (2013, 2016a, 2016b) showed that the Shapley value and the weighted Shapley value can be usefully applied to solving the problem of sharing joint liability among defendants especially in the case of sequential acts.[4]

Following the line of the literature, we take a cooperative game theory approach. In case that an indivisible value or a cost is generated jointly by a group of players, many solution concepts (allocation rules) have been proposed. The Shapley value is one of the most widely used allocation rules which ensures static efficiency.[5] In fact, it is proved that the Shapley value is the unique allocation rule satisfying static efficiency, symmetry, dummy and additivity.[6]

In determining the apportionment of damages among multiple defendants, the Shapley value may appear to be fair in the sense that it satisfies symmetry requiring that if two players contribute the same to any coalition, their share should be the same. In this paper, however, we argue that it is not strictly fair in the following sense. Suppose that an accident may occur only if the total sum of the defendant's negligence meets the threshold of 10. If the negligence of one defendant is 9 and the other's negligence is 1, each defendant should be liable for one half of the total damage, because neither could have caused the accident alone.[7] Considering the feature that they would be equally liable even if the first defendant

[4] They used a peer group game by Branzei *et al.* (2002) within which a hierarchy representing a causation relation is built-in. Their model is more general in the sense that it encompasses the simultaneous case which corresponds to the case of a one-tier hierarchy.

[5] In this context, static efficiency means that the sum of allocations to each individual (or compensations of each individual) must be the same as the total surplus (total damage). This property is built in the definition of the allocation rule.

[6] See Shapley (1953) for the original definition and Hart (1989) for a survey on the concept and its applications.

[7] Stapleton (2013) defines a threshold case rather vaguely but quite generally. She defines it by a case in which "an injury occurrence requires a certain amount of an element, but does not require more and is not affected if there is more." (p. 41) So, it includes the cases that each of multiple independent tortious acts is a sufficient condition for the harm, (e.g., two independent fires were both large enough to destroy a property), and that neither of the individual independent acts is sufficient but both are necessary to reach the threshold, and that none of the acts is sufficient nor necessary to reach the threshold. In our example, negligence of both defendants is necessary to reach the threshold because both satisfy a but-for test, in other

was much more negligent, one could hardly say that the Shapley value allocation is fair in this case. This problem occurs in cooperative games with a threshold level, i.e., a game in which a surplus or a cost occurs only if the sum of individual contributions in a coalition exceeds a certain threshold level.

To correct the problem, we add two ingredients explicitly into the structure of the conventional cooperative game; negligence and uncertainty. First, we consider the negligence of each individual explicitly, while it was only implicitly considered in Dehez and Ferey (2013). Second, accidents usually do not occur deterministically but probabilistically. A particular action of tortfeasors may or may not incur damages. Therefore, uncertainty is, in nature, inherent in tort cases but most of the literature using a cooperative game theory approach has assumed it away mainly for simplicity. Based on this, we consider a stochastic cooperative game with a threshold level. In the example described above, the threshold level of 10 units is not absolute in reality. It may be affected by random events. It could be either less than 10 or greater than 10 probabilistically. Also, the actual negligence may differ from the observed negligence due to observation or measurement errors. With those random factors, either defendant may be solely liable for the damage. If one takes unpredictable events (which always exist in reality) into account, the first defendant should pay a larger amount of compensation than the second defendant, because the probability that he is solely liable is higher.

In this paper, therefore, we propose a new allocation rule, what we call stochastic Shapley value. Roughly speaking, it is to apportion damage among defendants by their expected marginal contribution computed with respect to the conditional probability given that the accident has occurred. This concept can be applied to a class of stochastic cooperative games with a threshold value, which is quite vast. Consider, for example, a variation of the pollution case that was introduced in Ferey and Dehez (2016a). Two firms negligently pour toxic chemicals to a pond. Assume that there exists a threshold of 50 units above which chemicals are lethal for fish in the pond. Assume that firm A and firm B poured 40 units and 10 units respectively. The prediction by the Shapley value is that the two firms share the damage equally, even though firm A was more negligent. In this case, the court should take relative negligence seriously and this is consistent with the prediction by the stochastic Shapley value. The famous case of Paul's car is similar.[8] Consider its slight variation.

words, both are factual causations.

[8] The Paul's car case is illustrated in the Restatement (Third) of Torts as follows: Able, Baker and Charlie, acting independently but simultaneously, each negligently lean on Paul's car, which is parked at the lookout at the top of a mountain. Their combined force results in the car rolling over a diminutive curbstone

Suppose that A and B negligently lean on Paul's car parked at a scenic lookout of the top of a mountain. If it requires a certain threshold level of force, say 10 units, to push the car past the curbstone, and A and B pushed the car by the force of 7 units and 3 units respectively in the same direction, it is fair that A is more liable than B contrary to the Shapley value allocation. These examples suggest that the Shapley value does not reflect enough information about degrees of relative negligence. None of the papers take account of relative negligence explicitly in determining the apportionment rule. This paper addresses the issue of how the degree of fault of each defendant can affect the sharing rule when the probability of an accident depends on the negligence of each individual.

The stochastic Shapley value is to extend the concept of Shapley value to a stochastic setting in which the value is stochastically determined. There are other allocation rules that extend the Shapley value to a stochastic situation. For example, Charnes and Granot (1973) proposed the concept of the prior Shapley value.[9] But it is an *ex ante* concept.[10] If it is defined in *ex ante* terms, it cannot satisfy *ex post* efficiency, although it may satisfy *ex ante* efficiency. Then, it cannot be used as a proper rule of apportioning damages, because the sum of compensations may not cover the total amount of the damage that is actually incurred. The allocation based on the stochastic Shapley value is, however, *ex post* efficient and moreover feasible in the sense that each share is always nonnegative. It also satisfies fairness requiring that a more negligent defendant is more liable for the resulting damage.[11] Finally, we show under normal distribution that it converges to the Shapley value as the variances of random terms go to zero, as long as all involved defendants are negligent.

Our result that the damage should be apportioned not equally to each defendant, but based on their relative causal contributions to the plaintiff's injury corresponds with the case of Moore v. Johns-Mansville Sales Corp[12] which rejects pro-rata liability in favor of

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and plummeting down the mountain to its destruction. The force exerted by the push of any one actor would have been insufficient to propel Paul's car past the curbstone, but the combined force of any two of them is sufficient.

[9]Ma *et al.* (2008) is another example.

[10]Charnes and Granot (1977) and Granot (1977) further developed a two-stage solution whereby a prior payoff is promised in the first stage and it is adjusted to the realization of a random variable in the second stage.

[11]Van den Brink (2001) also proposed a modification of Shapley value satisfying fairness, but the definition of fairness is quite different. By his definition, an allocation rule is fair if, whenever a game is added, allocations of symmetric players in the game increase (or decrease) by the same amount. This definition of fairness is more or less closer to the fairness (or equal bargaining power) by Myerson (1977), rather than our definition.

[12] In this asbestos-related case, the Texas court rejected the contention that the defendants should bear

apportioned liability based upon relative causation.[13] To contrast the apportion based on our concept with the Shapley value allocation, consider another actual case of Parker vs. Bell Asbestos Mines, Ltd. The plaintiff sued for her husband's death from lung cancer. Her deceased husband, however, had been a heavy smoker for many years, so his death was due to asbestos exposure or smoking. The jury found that smoking contributed sixty percent, and inhaling asbestos forty percent, to his lung cancer. Accordingly, the court reduced total damages, from $214,000, to $85,600 to reflect the percent of causation attributable to asbestos exposure. This may serve as evidence that the stochastic Shapley value can be used in apportioning damages to each defendant to reflect their expected marginal contributions.

The possibility that random factors can have a real effect on an injury occurrence complicates the determination of causation as well as apportioning liability. As in the case of Summers v. Tice, the but-for test can be ineffective. In fact, the but-for test provides a too weak causation criterion, because passing the test is just a necessary condition for the occurrence of the injury and so it may include some necessary but remote causes. In other cases, which are often called overdetermined causation cases, each of multiple factors would have been sufficient to produce the injury, but none of them was necessary for the injury. However, the requirement that each factor should have been sufficient by itself seems too restrictive. So, Hart and Hanoré (1959) and later Wright (1985) proposed an alternative causation criterion, so-called NESS (Necessary Element of a Sufficient Set), in overdetermined cases. Under this criterion, an act is a cause of an injury if and only if it is a necessary element of a set of antecedent actual conditions that is sufficient for the occurrence of the injury. All of these criteria could be clearly applied to deterministic cases, but once some stochastic element can be a factor to the consequence, it becomes obscure whether a particular act is necessary or sufficient to the consequence. For example, in the modified case of Paul's car with two defendants, suppose the threshold level of force is 10 units, and A and B push the car with the force of 5 units respectively in the same direction. Then, neither of the negligence is sufficient to the consequence but both are necessary for it. However, if wind might be blowing possibly from the opposite direction, neither may be necessary. Or, if A and B push the car with the force of 12 units respectively simultaneously in the same direction, both are sufficient but neither is necessary. However, if one takes account of the possibility of wind force, neither may be sufficient.

The paper is organized as follows. In Section 2, we illustrate the problem of the Shapley

the liability pro rata even though the degree of relative causation may not have been established scientifically.

[13] The pro rata contribution rule which is used, for example, under the current Massachusetts tort law, has been challenged by many legal scholars and practitioners. See Baltay (2001).

value in a simple deterministic model. In Section 3, we introduce the stochastic Shapley value in a general stochastic model. In Section 4, we propose an alternative allocation rule, what we call normalized Shapley value and compare it with the stochastic Shapley value. In Section 5, we apply the Stochastic Shapley value to an alternative model with measurement errors. Section 6 contains concluding remarks. All the proofs are in the Appendix.

2 Illustrative Model

For an illustrative purpose, we consider a simple deterministic model. A plaintiff (P) can suffer damage from two defendants, D_1 and D_2. Let x_i be the negligence of D_i. We assume that if $x_1 + x_2 \geq z$, an accident occurs with certainty and incurs damage d to P, where z is the threshold level of negligence that triggers an accident. This is what Stapleton calls a threshold case. In this paper, we only consider threshold cases.

Since the two defendants are jointly liable for the damage, we can use an approach of the cooperative game theory to determine how to share compensation to the plaintiff. For simplicity, we normalize d to one.

This situation can be considered as a transferrable utility game $\langle N, v\rangle$ where $N = \{1, 2, \cdots, n\}$ is the set of players ($n = 2$) and $v(S)$ is the characteristic function for any coalition $S \subset N$. The associated characteristic function is then defined by

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} x_i \geq z \\ 0 & \text{otherwise .} \end{cases} \tag{1}$$

As an allocation rule, we can use the Shapley value. It is defined by the average marginal contribution of player i, i.e.,

$$\varphi_i = \frac{1}{n!} \sum_{\pi \in \Pi_N} \Delta_i(S_i(\pi)), \tag{2}$$

where Π_N is the set of all permutations of N, $S_i(\pi)$ is the set of players preceding i in π and $\Delta_i(S) = v(S \cup \{i\}) - v(S)$ is a marginal contribution of player i to coalition S.

If the damage amount is apportioned between the defendants according to the Shapley value, it is computed as

$$\varphi_1 = \frac{1}{2}[v(1) - v(\emptyset)] + \frac{1}{2}[v(1, 2) - v(2)] = \frac{1}{2} \tag{3}$$

if $x_i \in (0, z)$, $i = 1, 2$ and $x_1 + x_2 \geq z$.

Similarly, if $x_i \geq z$ for $i = 1, 2$, $v(i) = v(N) = 1$ for $i = 1, 2$, so we have

$$\varphi_1 = \frac{1}{2}[v(1) - v(\emptyset)] + \frac{1}{2}[v(1,2) - v(2)] = \frac{1}{2} \tag{4}$$

for $x_i \geq z$, $i = 1, 2$.

Finally, if $x_i \geq z$ and $x_j < z$ for $j \neq i$,

$$\varphi_i = \frac{1}{2}[v(i) - v(\emptyset)] + \frac{1}{2}[v(1,2) - v(j)] = 1, \tag{5}$$

while $\varphi_j = 0$.

To summarize, we have

$$\varphi_i = \begin{cases} \frac{1}{2} & \text{if } x_1 + x_2 \geq z, 0 < x_1, x_2 < z \\ \frac{1}{2} & \text{if } x_1, x_2 \geq z \\ 1 & \text{if } x_i \geq z, x_j < z, j \neq i \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

It is interesting to note that for any asymmetric (x_1, x_2) with $x_1 \neq x_2 < z$ and $x_1 + x_2 \geq z$, the jointly liable defendants equally share the damage as $(\varphi_1, \varphi_2) = (\frac{1}{2}, \frac{1}{2})$ regardless of his relative negligence. (See Figure 1.) This is the case even when $x_1 = \epsilon$ and $x_2 = z - \epsilon$ for some small $\epsilon > 0$. This implies that the allocation rule based on Shapley value is not compatible with the contribution rule or the proportionate rule whereby multiple tortfeasors are liable for the damage in proportion to their relative negligence.[14]

3 General Stochastic Model

In this section, we consider a general stochastic model. Suppose multiple potential defendants D_i, $i = 1, 2, \cdots, n$ may inflict damage to P. An accident occurs with some probability that depends on $\sum_{i=1}^n x_i$. That is, an accident may not occur when $\sum_{i=1}^n x_i \geq z$ and, similarly, an accident may occur even when $\sum_{i=1}^n x_i < z$. More specifically, an accident occurs if $\sum_{i=1}^n y_i \geq z$ where $y_i = x_i + \epsilon_i$. Here, ϵ_i is an idiosyncratic random term and accordingly y_i is the actual contribution of D_i to the occurrence of an accident. In the example of Paul's car, ϵ_i could be interpreted as a sum of all external random forces including wind force. We assume that the random error term ϵ_i is i.i.d. and follows a normal distribution with mean

[14] Most U.S. states adopt a conditional proportionate rule whereby defendants are responsible for an amount equal to their percentage of fault unless the plaintiff is found to be more than 50% or 51% responsible for the accident.

zero and variance σ^2.[15] Since ϵ_i is a sum of all external idiosyncratic shocks, it is natural to assume that the distribution does not depend on the care levels which are determined by the defendants. The damage amount d is again normalized to one. Note that the damage amount does not depend on the care levels. This is to make our analysis focus on threshold cases.[16]

We begin our analysis by defining a stochastic cooperative game à la Ma *et al.* (2008) which basically adapts Suijs and Borm (1999).

Definition 1 *A stochastic cooperative game is defined by* $G = \langle N, \{X_S\}_{S \subset N} \rangle$ *where* N *is the set of players and* $X_S \in L^1(\mathbb{R})$ *is the stochastic characteristic function assigning to a coalition* S *a stochastic payoff with finite expectation.*

A stochastic cooperative game is distinguished from a standard cooperative game only in that the value of the characteristic function is stochastic.

In our model, the stochastic characteristic function is given by

$$X_S = \begin{cases} 1 & \text{if } \sum_{i \in S} y_i \geq z \\ 0 & \text{otherwise .} \end{cases} \tag{7}$$

It is convenient to note that X_N is a random variable such that

$$X_N = t = \begin{cases} 1 & \text{if } \sum_{i \in N} y_i \geq z \\ 0 & \text{otherwise .} \end{cases} \tag{8}$$

We now define a stochastic Shapley value by modifying the definition of the extended Shapley value proposed by Ma *et al.* (2008).

Definition 2 *For a stochastic cooperative game* $G = \langle N, \{X_S\}_{S \subset N} \rangle$*, the stochastic Shapley value of player* i *is defined by*

$$\varphi_i(G, t) = \frac{1}{n!} \sum_{\pi \in \Pi_N} \mathbb{E}\left[\Delta_i(S_i(\pi)) | t\right], \tag{9}$$

where $\Delta_i(S) = X_{S \cup \{i\}} - X_S$ *for* $i \notin S$.

[15] We assume a normal distribution of the error term, because it is well known that the normal distribution approximates many natural phenomenon including measurement errors. See Wikipedia Contributors (2019).

[16] According to the definition of Stapleton (2013), more of a causal element than the threshold does not affect an injury in a threshold case.

Although we take expectations in the definition of φ_i, the stochastic Shapley value is not an *ex ante* concept. It is an *ex post* concept in the sense that it is the average marginal contribution on the condition that $t = 0, 1$ is realized, i.e., the accident has occurred or not. Therefore, it is distinguished from the prior Shapley value by Charnes and Granot (1973) which is defined in an *ex ante* term.

Given $x = (x_1, x_2, \cdots, x_n)$, let $\mathcal{G}_x = \{G_x \mid \epsilon_i \sim N(0, \sigma^2), \forall i \in N\}$ and consider any allocation rule ϕ defined on $\mathcal{G}_x \times \{0, 1\}$, i.e., $\phi : \mathcal{G}_x \times \{0, 1\} \to \mathbb{R}^n$. A list of axioms for the allocation rule are in order.

Definition 3 (Efficiency) *An allocation rule ϕ is (*ex post*) efficient if $\sum_{i \in N} \phi_i(G_x, t) = t$ for all $G_x \in \mathcal{G}_x$ and for all t.*

Definition 4 (Symmetry) *An allocation rule ϕ is symmetric if $\phi_i(G_x, t) = \phi_j(G_x, t)$ whenever i and j are* substitutes, *i.e., $X_{S \cup \{i\}} = X_{S \cup \{j\}}$ for any $S \subset N$ with $i, j \notin S$.*

Definition 5 (Dummy) *An allocation rule ϕ satisfies dummy if $\phi_i(G_x, t) = 0$ for any null player i, i.e., $X_{S \cup \{i\}} = X_S$ for any $(i \notin) S \subset N$.*

We will suppress G_x if there is no chance of confusion. Besides the standard axioms described above, we are interested in the following additional axioms.

Definition 6 *An allocation rule ϕ is feasible if $0 \leq \phi_i(t) \leq t$, for all $i \in N$, for all t.*

Definition 7 *An allocation rule ϕ is fair (monotonic) if $\phi_i \geq \phi_j$ whenever $x_i \geq x_j$ for $i \neq j$.*

Definition 8 *An allocation rule ϕ satisfies convergence if $\lim_{\sigma \to 0} \phi_i = \frac{1}{n}$.*

Note that the axiom of symmetry implies that $\phi_i = \phi_j$ if $x_i = x_j$,[17] but does not imply that $\phi_i > \phi_j$ if $x_i > x_j$. The axiom of convergence says that the stochastic Shapley value should converge to the deterministic Shapley value, $(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$, as the variance of ϵ_i goes to zero.

It may help get some insights for the axioms to consider a simple case of $n = 2$.

[17]This is because $\mathbb{E}[X_{S \cup \{i\}} \mid t] = \mathbb{P}(y_i + \sum_{k \in S} y_k \geq z \mid t) = \mathbb{P}(y_i \geq z - \sum_{k \in S} y_k \mid t) = \mathbb{P}(\epsilon_i \geq z - \sum_{k \in S} y_k - x_i \mid t)$, $\mathbb{E}[X_{S \cup \{j\}} \mid t] = \mathbb{P}(y_j \geq z - \sum_{k \in S} y_k \mid t) = \mathbb{P}(\epsilon_j \geq z - \sum_{k \in S} y_k - x_i \mid t)$ if $x_i = x_j$, and thus $\mathbb{E}[X_{S \cup \{i\}} \mid t] = \mathbb{E}[X_{S \cup \{j\}} \mid t]$, i.e., i and j are substitutes, which implies that $\phi_i = \phi_j$ by the axiom of symmetry.

Efficiency Since

$$\begin{aligned}\mathbb{E}[X_S|t=1] &= \mathbb{P}[\sum_{i\in S} y_i \geq z | \sum_{i\in N} y_i \geq z] \\ &= \mathbb{P}[\sum_{i\in S} \epsilon_i \geq z - \sum_{i\in S} x_i | \sum_{i\in N} \epsilon_i \geq z - \sum_{i\in N} x_i], \end{aligned} \tag{10}$$

the stochastic Shapley value is computed as

$$\begin{aligned}\varphi_1(t=1) &= \frac{1}{2}\mathbb{E}[\Delta_1(\emptyset)|t=1] + \frac{1}{2}\mathbb{E}[\Delta_1(\{2\})|t=1] \\ &= \frac{1}{2}\mathbb{E}[X_{\{1\}}|t=1] + \frac{1}{2}(\mathbb{E}[X_{\{1,2\}}|t=1] - \mathbb{E}[X_{\{2\}}|t=1]) \\ &= \frac{1}{2} - \frac{1}{2}\mathbb{P}[\epsilon_2 \geq z - x_2 | \epsilon_1 + \epsilon_2 \geq z - (x_1 + x_2)] \\ &\quad + \frac{1}{2}\mathbb{P}[\epsilon_1 \geq z - x_1 | \epsilon_1 + \epsilon_2 \geq z - (x_1 + x_2)] \\ &= \frac{1}{2} + \frac{1}{2}\mathbb{P}[z - x_1 \leq \epsilon_1 \leq z - x_2 | \epsilon_1 + \epsilon_2 \geq z - (x_1 + x_2)] \qquad (\because \text{i.i.d}) \\ &= \frac{1}{2} + \frac{1}{2}\Delta^C, \qquad (11) \\ \varphi_2(t=1) &= \frac{1}{2} - \frac{1}{2}\Delta^C, \qquad (12)\end{aligned}$$

where

$$\Delta^C := \frac{\mathbb{P}[z - x_1 \leq \epsilon_1 \leq z - x_2 \wedge \epsilon_1 + \epsilon_2 \geq z - (x_1 + x_2)]}{\mathbb{P}[\epsilon_1 + \epsilon_2 \geq z - (x_1 + x_2)]}.$$

From (11) and (12), we can directly check that $\varphi_1(t=1) + \varphi_2(t=1) = 1$. This implies that (φ_1, φ_2) satisfies *ex post* efficiency for the case that $t = 1$. It is not difficult to see that $\varphi_1(t=0) + \varphi_2(t=0) = 0$, i.e., *ex post* efficiency is satisfied.[18] Hereafter, we will simply write $\varphi_i(t=1)$ as φ_i, since the case that $t = 0$ (the case of no accident) is not in our interest.

Feasibility Equations (11) and (12) imply that $\varphi_i \in [0,1]$ for $i = 1, 2$, since $\Delta^C \leq 1$. That is, the allocation according to the stochastic Shapley value is feasible.

Fairness (Monotonicity) In equations (11) and (12), Δ^C is a difference in conditional probabilities that D_1 is solely liable and that D_2 is solely liable conditional on the occurrence of an accident (i.e, $y_1 + y_2 \geq z$). It is clear that $0 \leq \Delta^C \leq 1$, as long as $x_1 \geq x_2$. Since

[18]See Proposition 1 for the proof.

$\Delta^C \geq 0$, we have $\varphi_1 \geq \varphi_2$, if and only if $x_1 \geq x_2$, i.e., the stochastic Shapley value satisfies fairness (monotonicity). Contrary to the deterministic case, the stochastic Shapley value implies that a defendant who is more negligent should compensate the plaintiff more. That is, the stochastic Shapley value reflects the negligence of each defendant when there are two defendants.

Convergence If $x_1 + x_2 = z$, we have

$$\mathbb{P}[\epsilon_1 + \epsilon_2 \geq z - (x_1 + x_2)] = \mathbb{P}[\epsilon_1 + \epsilon_2 \geq 0] = \frac{1}{2}$$

since $\mathbb{P}[\epsilon_1 + \epsilon_2 \geq 0] = \mathbb{P}[\epsilon_1 + \epsilon_2 \leq 0] = \frac{1}{2}$ due to the symmetry of $f(\epsilon)$. Also, since ϵ follows a normal distribution with mean zero and variance σ^2, we have

$$f(\epsilon) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\epsilon^2}{2\sigma^2}}.$$

Lemma 1 $\mathbb{P}[\epsilon \geq \alpha] = \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{\alpha}{\sqrt{2}\sigma}\right)\right)$, *where* $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, $\operatorname{erf}(0) = 0$ *and* $\lim_{x \to \infty} \operatorname{erf}(x) = 1$.

In the Appendix, we provide a proof for a more generalized version of Lemma 1.

For $x_i > 0$, $(\forall i \in N)$, using this lemma with the symmetry of $f(\epsilon)$, we have

$$\begin{aligned}
&\mathbb{P}[x_2 \leq \epsilon_1 \leq x_1 \wedge \epsilon_1 + \epsilon_2 \geq 0] \\
&= \frac{1}{2}\left(\mathbb{P}[-x_1 \leq \epsilon_2 \leq -x_2 \ \wedge \epsilon_1 + \epsilon_2 \leq 0] + \mathbb{P}[x_2 \leq \epsilon_1 \leq x_1 \wedge \epsilon_1 + \epsilon_2 \geq 0]\right) \\
&= \frac{1}{2}\left(\mathbb{P}[\epsilon_1 \leq x_1 \wedge \epsilon_2 \geq -x_1] - \mathbb{P}[\epsilon_1 \leq x_2 \ \wedge \ \epsilon_2 \geq -x_2]\right) \\
&= \frac{1}{2}\left(\mathbb{P}[\epsilon_1 \leq x_1]\mathbb{P}[\epsilon_2 \geq -x_1] - \mathbb{P}[\epsilon_1 \leq x_2]\mathbb{P}[\epsilon_2 \geq -x_2]\right) \\
&= \frac{1}{2}\left(\frac{1}{4}\left(1 + \operatorname{erf}\left(\frac{x_1}{\sqrt{2}\sigma}\right)\right)^2 - \frac{1}{4}\left(1 + \operatorname{erf}\left(\frac{x_2}{\sqrt{2}\sigma}\right)\right)^2\right) \\
&= \frac{1}{8}\left(\left(1 + \operatorname{erf}\left(\frac{x_1}{\sqrt{2}\sigma}\right)\right)^2 - \left(1 + \operatorname{erf}\left(\frac{x_2}{\sqrt{2}\sigma}\right)\right)^2\right) \qquad (13)
\end{aligned}$$

and

$$\varphi_1 = \frac{1}{2} + \frac{1}{8}\left(\left(1 + \operatorname{erf}\left(\frac{x_1}{\sqrt{2}\sigma}\right)\right)^2 - \left(1 + \operatorname{erf}\left(\frac{x_2}{\sqrt{2}\sigma}\right)\right)^2\right), \qquad (14)$$

$$\varphi_2 = \frac{1}{2} - \frac{1}{8}\left(\left(1 + \operatorname{erf}\left(\frac{x_1}{\sqrt{2}\sigma}\right)\right)^2 - \left(1 + \operatorname{erf}\left(\frac{x_2}{\sqrt{2}\sigma}\right)\right)^2\right). \qquad (15)$$

Therefore, (φ_1, φ_2) converges to $(\frac{1}{2}, \frac{1}{2})$ as σ goes to zero, since $\lim_{\sigma \to 0} \mathrm{erf}(\frac{x_i}{\sqrt{2}\sigma}) = 1$. That is, as the effect of other factors than the defendants' negligence becomes smaller, (i.e., σ goes to zero), the stochastic Shapley value converges to the deterministic Shapley value.

If $x_1 = z > 0$ and $x_2 = 0$, however, we have

$$\varphi_1 = \frac{1}{2} + \frac{1}{8}\left(\left(1 + \mathrm{erf}\left(\frac{x_1}{\sqrt{2}\sigma}\right)\right)^2 - 1\right), \tag{16}$$

$$\varphi_2 = \frac{1}{2} - \frac{1}{8}\left(\left(1 + \mathrm{erf}\left(\frac{x_1}{\sqrt{2}\sigma}\right)\right)^2 - 1\right). \tag{17}$$

Then, as $\sigma \to 0$, $(\varphi_1, \varphi_2) \to (7/8, 1/8)$ which is different from the deterministic Shapley value. It is not surprising that a non-negligent player ($x_i = 0$) pays a positive compensation, because there is a small probability that even a non-negligent defendant is solely liable for the damage, unless $x_i = 0$ implies $\epsilon_i = 0$. That is, the convergence of the stochastic Shapley value is guaranteed only if $x_i > 0$ for all i.

Now, we have the following general proposition.

Proposition 1 *The stochastic Shapley value $(\varphi_i(t))_{i \in N}$ satisfies* ex post *efficiency, symmetry, dummy, feasibility, fairness and convergence for any $x_i \in (0, 1)$, for any $i \in N$, for any t.*

Unfortunately, however, the stochastic Shapley value is not the unique allocation rule satisfying efficiency, symmetry and dummy. In the next section, we will introduce another allocation rule satisfying all the properties and even fairness.

4 Normalized Shapley Value

We propose an alternative allocation rule satisfying efficiency, fairness and convergence, what we call the normalized Shapley value.

Definition 9 *For a stochastic cooperative game $G = \langle N, \{X_S\}_{S \subset N} \rangle$, the normalized Shapley value of player i is defined by*

$$\phi_i(G, t) = \frac{1}{n! \mathbb{P}[X_N = t]} \sum_{\pi \in \Pi_N} \mathbb{E}\left[\Delta_i(S_i(\pi))\right], \tag{18}$$

where $\Delta_i(S) = X_{S \cup \{i\}} - X_S$ for $i \notin S$.

The idea of this allocation rule is to normalize the allocation rule (obtained without using conditional probability) so as to satisfy the efficiency condition. Thus, the unconditional expectation of a characteristic function is

$$\begin{aligned}\mathbb{E}[X_S] &= \mathbb{P}[\sum_{i\in S} y_i \geq z] \\ &= \mathbb{P}[\sum_{i\in S} \epsilon_i \geq z - \sum_{i\in S} x_i]. \end{aligned} \tag{19}$$

For $n = 2$, we have

$$\begin{aligned}\phi_1 &= \frac{1}{2\mathbb{P}[X_N = 1]}\mathbb{E}[\Delta_1(\emptyset)] + \frac{1}{2\mathbb{P}[X_N = 1]}\mathbb{E}[\Delta_1(\{2\})] \\ &= \frac{1}{2\mathbb{P}[X_N = 1]}\mathbb{E}[X_{\{1\}}] + \frac{1}{2\mathbb{P}[X_N = 1]}(\mathbb{E}[X_{\{1,2\}}] - \mathbb{E}[X_{\{2\}}]) \\ &= \frac{\mathbb{P}[\epsilon_1 \geq z - x_1]}{2\mathbb{P}[\epsilon_1 + \epsilon_2 \geq z - (x_1 + x_2)]} + \frac{\mathbb{P}[\epsilon_1 + \epsilon_2 \geq z - (x_1 + x_2)]}{2\mathbb{P}[\epsilon_1 + \epsilon_2 \geq z - (x_1 + x_2)]} - \frac{\mathbb{P}[\epsilon_2 \geq z - x_2]}{2\mathbb{P}[\epsilon_1 + \epsilon_2 \geq z - (x_1 + x_2)]} \\ &= \frac{1}{2} + \frac{\mathbb{P}[z - x_1 \leq \epsilon \leq z - x_2]}{2\mathbb{P}[\epsilon_1 + \epsilon_2 \geq z - (x_1 + x_2)]}, \qquad (20) \\ \phi_2 &= \frac{1}{2} - \frac{\mathbb{P}[z - x_1 \leq \epsilon \leq z - x_2]}{2\mathbb{P}[\epsilon_1 + \epsilon_2 \geq z - (x_1 + x_2)]}. \qquad (21)\end{aligned}$$

In particular, if $x_1 + x_2 = z$, we have

$$\frac{1}{2}\mathbb{P}[\epsilon_1 + \epsilon_2 \geq z - (x_1 + x_2)] = \frac{1}{2}\mathbb{P}[\epsilon_1 + \epsilon_2 \geq 0] = \frac{1}{4}.$$

Due to Lemma 1, we have

$$\phi_1 = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x_1}{\sqrt{2}\sigma}\right) - \mathrm{erf}\left(\frac{x_2}{\sqrt{2}\sigma}\right)\right), \tag{22}$$

$$\phi_2 = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x_2}{\sqrt{2}\sigma}\right) - \mathrm{erf}\left(\frac{x_1}{\sqrt{2}\sigma}\right)\right). \tag{23}$$

It is clear that (ϕ_1, ϕ_2) satisfies efficiency. Also, it satisfies fairness because $\phi_1 \geq \phi_2$ if and only if $x_1 \geq x_2$. Finally, we can show that as σ goes to zero, the normalized stochastic Shapley value converges to the Shapley value. If $x_1, x_2 > 0$, $(\phi_1, \phi_2) \to (1/2, 1/2)$, since $\lim_{\sigma\to 0} \mathrm{erf}\left(\frac{x_1}{\sqrt{2}\sigma}\right) = \lim_{\sigma\to 0} \mathrm{erf}\left(\frac{x_2}{\sqrt{2}\sigma}\right) = 1$.[19]

[19] Convergence of the normalized Shapley value holds even for $x_i = 0$. See the appendix for the proof for the general convergence property of the normalized Shapley value. (Claim 1)

A drawback of the normalized Shapley value is that the allocations may not be feasible. To see this, consider

$$\Delta^N := \frac{\mathbb{P}[z - x_1 \le \epsilon \le z - x_2]}{\mathbb{P}[\epsilon_1 + \epsilon_2 \ge z - (x_1 + x_2)]}$$

as a counterpart for Δ^C. It is not difficult to observe the possibility that $\Delta^N > 1$,[20] so that $\phi_2 = \frac{1}{2} - \frac{1}{2}\Delta^N < 0$, since it is not a conditional probability. This possibility does not occur in the stochastic Shapley value because it is a conditional probability. The main difference between the stochastic Shapley value and the normalized Shapley value, therefore, lies simply in whether we use conditional expectations or unconditional expectations.

5 Stochastic Shapley Value in a Model with Measurement Errors

So far, we assumed that uncertainty comes in the process that an act causes a damage. In this section, we consider an alternative model in which uncertainty comes in a form of measurement errors.

Again, we focus only on threshold cases, so assume that an accident occurs if $\sum_{i=1}^n x_i \ge z$ where x_i denotes the actual value of negligence of D_i. Let y_i denote the measured value for x_i.[21] Since the judge can only observe y_i, not x_i, the judicial decision must be based on y_i, not on x_i.

The difference between x_i and y_i is a measurement error which is denoted by ϵ_i. So, we have $y_i = x_i + \epsilon_i$. We assume that ϵ_i is i.i.d. and follows a normal distribution with mean zero and variance σ^2. The damage amount is again normalized to one.

Below, we briefly show that the stochastic Shapley value preserves the main properties of efficiency, feasibility, fairness and convergence in this model of measurement errors.

Efficiency Since

$$\begin{aligned}\mathbb{E}[X_S|t=1] &= \mathbb{P}[\sum_{i\in S} x_i \ge z | \sum_{i\in N} x_i \ge z] \\ &= \mathbb{P}[\sum_{i\in S} \epsilon_i \le \sum_{i\in S} y_i - z | \sum_{i\in N} \epsilon_i \le \sum_{i\in N} y_i - z], \end{aligned} \tag{24}$$

the stochastic Shapley value is computed as

[20] A numerical example is provided in the Appendix.
[21] We are abusing notation here.

$$
\begin{aligned}
\varphi_1(t=1) &= \frac{1}{2}\mathbb{E}[\Delta_1(\emptyset)|t=1] + \frac{1}{2}\mathbb{E}[\Delta_1(\{2\})|t=1] \\
&= \frac{1}{2}\mathbb{E}[X_{\{1\}}|t=1] + \frac{1}{2}(\mathbb{E}[X_{\{1,2\}}|t=1] - \mathbb{E}[X_{\{2\}}|t=1]) \\
&= \frac{1}{2} - \frac{1}{2}\mathbb{P}[\epsilon_2 \leq y_2 - z|\epsilon_1 + \epsilon_2 \leq (y_1+y_2) - z] \\
&\quad + \frac{1}{2}\mathbb{P}[\epsilon_1 \leq y_1 - z|\epsilon_1 + \epsilon_2 \leq (y_1+y_2) - z] \\
&= \frac{1}{2} + \frac{1}{2}\mathbb{P}[y_2 - z \leq \epsilon_1 \leq y_1 - z|\epsilon_1 + \epsilon_2 \leq (y_1+y_2) - z] \\
&= \frac{1}{2} + \frac{1}{2}\Delta^M, \qquad (25) \\
\varphi_2(t=1) &= \frac{1}{2} - \frac{1}{2}\Delta^M, \qquad (26)
\end{aligned}
$$

where

$$
\Delta^M := \frac{\mathbb{P}[y_2 - z \leq \epsilon_1 \leq y_1 - z \wedge \epsilon_1 + \epsilon_2 \leq (y_1+y_2) - z]}{\mathbb{P}[\epsilon_1 + \epsilon_2 \leq (y_1+y_2) - z]}.
$$

Since $\varphi_1(t=1) + \varphi_2(t=1) = 1$, it is clear that (φ_1, φ_2) satisfies *ex post* efficiency for the case that $t = 1$.

Feasibility Feasibility is also clear from equations (25) and (26), because $\Delta^M \leq 1$ imply that $\varphi_i \in [0,1]$ for $i = 1,2$.

Fairness (Monotonicity) Fairness also follows directly from $\Delta^M > 0$ as long as $y_1 > y_2$. Measurement errors allow the possibility that $y_1 < y_2$ even if $x_1 > x_2$. Note that all that matters in determining shares of the damage is y_i, not x_i, since x_i is not observable. Therefore, our allocation rule is fair in the sense that $\varphi_1 \geq \varphi_2$, if and only if $y_1 \geq y_2$. Note that $x_1 \geq x_2$ does not necessarily imply $\varphi_1 \geq \varphi_2$.

Convergence For $y_i > 0$, $(\forall i \in N)$, we have

$$
\varphi_1 = \frac{1}{2} + \frac{1}{8}\left(\left(1 + \operatorname{erf}\left(\frac{y_1}{\sqrt{2}\sigma}\right)\right)^2 - \left(1 + \operatorname{erf}\left(\frac{y_2}{\sqrt{2}\sigma}\right)\right)^2\right), \qquad (27)
$$

$$
\varphi_2 = \frac{1}{2} - \frac{1}{8}\left(\left(1 + \operatorname{erf}\left(\frac{y_1}{\sqrt{2}\sigma}\right)\right)^2 - \left(1 + \operatorname{erf}\left(\frac{y_2}{\sqrt{2}\sigma}\right)\right)^2\right). \qquad (28)
$$

Therefore, as measurement errors tend to be very small (i.e., σ goes to zero), (φ_1, φ_2) converges to $(\frac{1}{2}, \frac{1}{2})$, which is the deterministic Shapley value, since $\lim_{\sigma\to 0} \operatorname{erf}(\frac{y_i}{\sqrt{2}\sigma}) = 1$.

6 Conclusion

In this paper, we proposed the concept of the stochastic Shapley value to resolve the puzzle we face in sharing damages in joint liability cases. We argued that the allocation rule based on the Shapley value is not fair in the sense that it does not, in general, reflect the relative negligence among defendants, but that the stochastic Shapley value yields fair allocations by capturing the possibility of random disturbances.

Considering the universal reality that a random factor is important in many accident occurrences, we believe that our concept of the stochastic Shapley value has a good potential of being applied to actual legal cases. For instance, there are a large amount of litigation on asbestos-related diseases and deaths. What makes litigation on asbestos difficult is to prove causation. Above all, those who develop asbestos-related diseases show no signs of illness for a long time after asbestos exposure. It can take from 10 to 40 years or more for symptoms of an asbestos-related condition to appear. Thus, during the latency period, the victim could work in several workplaces and other factors than asbestos exposure could cause the disease. Our concept could be useful especially in such a situation where possible causes are multiple and uncertain.

Although our concept of stochastic Shapley value was developed in the context of damage (cost) sharing, it could be also applied to the problem of value sharing. For example, it can give a useful guideline for allocating the prize from obtaining a patent as a consequence of joint research to each research unit if the contribution of each research unit cannot be clearly determined. Or, generally, allocating the reward for the output of any team project would be a proper subject to which our concept is usefully applicable. We look forward to seeing its enriched applications in the future.

Appendix

Proof of Proposition 1:

Efficiency The proof is almost same as the proof that original Shapley value satisfies efficiency. First, choose $\pi = (1, 2, \ldots, n) \in \Pi_N$. For this permutation, we have

$$\begin{aligned}
\mathbb{E}[\Delta_1(S_1(\pi))|t] &= \mathbb{E}[X_{\{1\}}|X_N = t] - \mathbb{E}[X_\emptyset|X_N = t] \\
\mathbb{E}[\Delta_2(S_2(\pi))|t] &= \mathbb{E}[X_{\{1,2\}}|X_N = t] - \mathbb{E}[X_{\{1\}}|X_N = t] \\
&\vdots \\
\mathbb{E}[\Delta_n(S_n(\pi))|t] &= \mathbb{E}[X_{\{1,2,\ldots,n\}}|X_N = t] - \mathbb{E}[X_{\{1,2,\ldots,n-1\}}|X_N = t].
\end{aligned}$$

By summing up all of these equations, we get

$$\sum_{i \in N} \mathbb{E}[\Delta_i(S_i(\pi))|t] = \mathbb{E}[X_{\{1,2,\ldots,n\}}|X_N = t] - \mathbb{E}[X_\emptyset|X_N = t] = \mathbb{E}[X_N|X_N = t] = t.$$

Similarly, the above equation holds for any permutation $\pi \in \Pi_N$. Adding up all these terms for each π and dividing by $n!$, we get

$$\sum_{i \in N} \varphi_i(G, t) = \frac{1}{n!} \sum_{i \in N} \sum_{\pi \in \Pi_N} \mathbb{E}[\Delta_i(S_i(\pi))|t] = \frac{1}{n!} \sum_{\pi \in \Pi_N} \sum_{i \in N} \mathbb{E}[\Delta_i(S_i(\pi))|t] = \frac{1}{n!} \sum_{\pi \in \Pi_N} t = t. \|$$

Symmetry For $\pi \in \Pi_N$, let π' be a permutation of N interchanging i and j for π. This gives a one-to-one correspondence from Π_N to itself. It is enough to show that $\mathbb{E}[\Delta_i(S_i(\pi))|t] = \mathbb{E}[\Delta_j(S_j(\pi'))|t]$. (If this holds, then we get $\varphi_i(G, t) = \varphi_j(G, t)$ by summing up all of these over $\pi \in \Pi_N$ ($\pi' \in \Pi_N$).)

If i precedes j in π, $S_i(\pi) = S_j(\pi') =: S$. Then $\mathbb{E}[X_{S \cup \{i\}}|X_N = t] = \mathbb{E}[X_{S \cup \{j\}}|X_N = t]$ and

$$\mathbb{E}[\Delta_i(S_i(\pi))|t] = \mathbb{E}[X_{S_i(\pi) \cup \{i\}}|t] - \mathbb{E}[X_{S_i(\pi)}|t] = \mathbb{E}[X_{S_j(\pi') \cup \{j\}}|t] - \mathbb{E}[X_{S_j(\pi')}|t] = \mathbb{E}[\Delta_j(S_j(\pi))|t].$$

If j precedes i, $j \in S_i(\pi)$. If we put $S' = S_i(\pi) - \{j\}$, then $S_i(\pi') - \{i\} = S'$ and

$$\begin{aligned}
&\mathbb{E}[\Delta_i(S_i(\pi))|t] = \mathbb{E}[X_{S_i(\pi) \cup \{i\}}|t] - \mathbb{E}[X_{S_i(\pi)}|t] = \mathbb{E}[X_{S' \cup \{i,j\}}|t] - \mathbb{E}[X_{S' \cup \{j\}}|t] \\
&= \mathbb{E}[X_{S' \cup \{i,j\}}|t] - \mathbb{E}[X_{S' \cup \{i\}}|t] = \mathbb{E}[X_{S_j(\pi') \cup \{j\}}|t] - \mathbb{E}[X_{S_j(\pi')}|t] = \mathbb{E}[\Delta_j(S_j(\pi'))|t].
\end{aligned}$$

In both cases, we have $\mathbb{E}[\Delta_i(S_i(\pi))|t] = \mathbb{E}[\Delta_j(S_j(\pi'))|t]$, which completes the proof. $\|$

Dummy Player Let i be a dummy player. By definition of a dummy player, we have $\mathbb{E}[\Delta_i(S_i(\pi))|t] = \mathbb{E}[X_{S_i(\pi)\cup\{i\}}|t] - \mathbb{E}[X_{S_i(\pi)}|t] = 0$, and thus $\varphi_i(G,t) = 0$. $\|$

Feasibility We have

$$\begin{aligned}
\mathbb{E}[\Delta_i(S) \mid t] &= \mathbb{E}[X_{S\cup\{i\}} - X_S \mid t] \\
&= \mathbb{P}(X_{S\cup\{i\}} = 1 \wedge X_S = 0 \mid t) - \mathbb{P}(X_{S\cup\{i\}} = 0 \wedge X_S = 1 \mid t) \\
&= \mathbb{P}(\sum_{j\in S} y_j < z \wedge \sum_{j\in S\cup\{i\}} y_j \geq z \mid t) - \mathbb{P}(\sum_{j\in S} y_j \geq z \wedge \sum_{j\in S\cup\{i\}} y_j < z \mid t) \\
&= \int_0^\infty h_{(k)}(\sum_{j=1}^k y_j = z - s \mid t)\mathbb{P}(y_i \geq s \mid t)ds \\
&\quad - \int_0^\infty h_{(k)}(\sum_{j=1}^k y_j = z + s \mid t)\mathbb{P}(y_i \leq -s \mid t)ds \\
&> 0,
\end{aligned}$$

for $i \notin S = \{y_1, \cdots, y_k\}$, where $h_{(k)}$ is the joint pdf of $y_1, \cdots, y_k$ conditional on t, since $h_{(k)}(z - s \mid t) > h_{(k)}(z + s \mid t)$ and $\mathbb{P}(y_i \geq s \mid t) > \mathbb{P}(y_i \leq -s \mid t)$ for any $s \geq 0$ due to $x_i > 0$ and the assumption of normal distributions. It is also clear that $\Delta_i(S) \leq 1$ for any S. Therefore, we have $\varphi_i(t = 1) = \frac{1}{n!}\sum_{\pi\in\Pi_N} \mathbb{E}\left[\Delta_i(S_i(\pi)) \mid t = 1\right] \in [0, 1]$. $\|$

Fairness (Monotonicity) For general n, the stochastic Shapley value of D_1 is

$$\begin{aligned}
\varphi_1 &= \sum_{S\subset N\setminus\{1\}} \frac{s!(n-1-s)!}{n!}\mathbb{E}[\Delta_1(S)|t = 1] \\
&= \sum_{S\subset N\setminus\{1\}} \frac{s!(n-1-s)!}{n!}\left(\mathbb{P}\left[\sum_{i\in S\cup\{1\}} \epsilon_i \geq z - \left(\sum_{i\in S\cup\{1\}} x_i\right) \middle| \sum_{i\in N} \epsilon_i \geq z - \left(\sum_{i\in N} x_i\right)\right]\right. \\
&\quad \left. -\mathbb{P}\left[\sum_{i\in S} \epsilon_i \geq z - \left(\sum_{i\in S} x_i\right) \middle| \sum_{i\in N} \epsilon_i \geq z - \left(\sum_{i\in N} x_i\right)\right]\right), \qquad (29)
\end{aligned}$$

where $s = |S|$. We can compute $\varphi_2, \cdots, \varphi_n$ similarly.

To show that $\varphi_i \geq \varphi_j$ if and only if $x_i \geq x_j$, we assume without loss of generality that

$x_1 \geq x_2 \geq \cdots \geq x_n$. We start by defining the following collections of coalitions

$$\begin{aligned} A &= \{S \subset N \backslash \{1\} \mid 2 \notin S\}, \\ B &= \{S \subset N \backslash \{1\} \mid 2 \in S\}, \\ A' &= \{S \subset N \backslash \{2\} \mid 1 \notin S\}, \\ B' &= \{S \subset N \backslash \{2\} \mid 1 \in S\}. \end{aligned}$$

The collections of coalitions A and B partition $N \setminus \{1\}$ in the sense that $A \cup B = 2^{N \setminus \{1\}}$ and $A \cap B = \emptyset$. Similarly, $\{A', B'\}$ is a partition of $N \setminus \{2\}$. Also, note that $A = \{S \subset N \setminus \{1, 2\}\} = A'$. Then, by using the sets, a stochastic Shapley value can be decomposed into two components

$$\begin{aligned} \varphi_1 &= \sum_{S \in A} \frac{s!(n-1-s)!}{n!} \mathbb{E}[\Delta_1(S) | t = 1] + \sum_{S \in B} \frac{s!(n-1-s)!}{n!} \mathbb{E}[\Delta_1(S) | t = 1] \\ &\equiv L_1 + M_1, \qquad (30) \\ \varphi_2 &= \sum_{S \in A'} \frac{s!(n-1-s)!}{n!} \mathbb{E}[\Delta_2(S) | t = 1] + \sum_{S \in B'} \frac{s!(n-1-s)!}{n!} \mathbb{E}[\Delta_2(S) | t = 1] \\ &\equiv L_2 + M_2. \qquad (31) \end{aligned}$$

First, consider $S \in A (= A')$. We have

$$\begin{aligned} & x_1 \geq x_2 \\ \Leftrightarrow \; & z - \left(\sum_{i \in S \cup \{1\}} x_i \right) \leq z - \left(\sum_{i \in S \cup \{2\}} x_i \right) \\ \Leftrightarrow \; & \mathbb{P}\left[\sum_{i \in S \cup \{1\}} \epsilon_i \geq z - \left(\sum_{i \in S \cup \{1\}} x_i \right) \middle| t \right] \geq \mathbb{P}\left[\sum_{i \in S \cup \{2\}} \epsilon_i \geq z - \left(\sum_{i \in S \cup \{2\}} x_i \right) \middle| t \right], \end{aligned}$$

which yields $L_1 \geq L_2$.

Now, consider $S \in B$. Since $2 \in S$, we can write S as $S = S_0 \cup \{2\}$ where $S_0 \subset N \setminus \{1, 2\}$. Similarly, we can define $S' \equiv S_0 \cup \{1\}$. Then, a mapping between S and S' is a one-to-one correspondence with the mapping between B and B'. Then, similarly, we have

$$\begin{aligned} & \mathbb{P}\left[\sum_{i \in S_0 \cup \{1,2\}} \epsilon_i \geq z - \left(\sum_{i \in S_0 \cup \{1,2\}} x_i \right) \middle| t \right] - \mathbb{P}\left[\sum_{i \in S_0 \cup \{2\}} \epsilon_i \geq z - \left(\sum_{i \in S_0 \cup \{2\}} x_i \right) \middle| t \right] \\ \geq \; & \mathbb{P}\left[\sum_{i \in S_0 \cup \{1,2\}} \epsilon_i \geq z - \left(\sum_{i \in S_0 \cup \{1,2\}} x_i \right) \middle| t \right] - \mathbb{P}\left[\sum_{i \in S_0 \cup \{1\}} \epsilon_i \geq z - \left(\sum_{i \in S_0 \cup \{1\}} x_i \right) \middle| t \right], \end{aligned}$$

yielding $M_1 \geq M_2$. This concludes that $\varphi_1 \geq \varphi_2$. $\|$

Convergence We will use the notation

$$p_S \equiv \mathbb{P}\left[\sum_{i\in S}\epsilon_i \geq z - \sum_{i\in S} x_i \middle| \sum_{i\in N}\epsilon_i \geq 0\right] = \mathbb{P}\left[\sum_{i\in S}\epsilon_i \geq \alpha_S \middle| \sum_{i\in N}\epsilon_i \geq 0\right] \tag{32}$$

where $\alpha_S \equiv z - \sum_{i\in S} x_i$ for convenience. Note that $\alpha_S > 0$ iff $S \neq N$. Then the stochastic Shapley value is given by

$$\varphi_i = \sum_{S\subset N\setminus\{i\}} \frac{s!(n-1-s)!}{n!}(p_{S\cup\{i\}} - p_S). \tag{33}$$

Our claim is that $\lim_{\sigma\to 0} p_S = 0$ for $S \subsetneq N$. We have

$$\begin{aligned}
0 \leq p_S &= \frac{\mathbb{P}\left[\sum_{i\in S}\epsilon_i \geq \alpha_S \wedge \epsilon_i \geq 0\right]}{\mathbb{P}\left[\sum_{i\in N}\epsilon_i \geq 0\right]} && (34)\\
&= 2\mathbb{P}\left[\sum_{i\in S}\epsilon_i \geq \alpha_S \wedge \epsilon_i \geq 0\right] && (35)\\
&\leq 2\mathbb{P}\left[\sum_{i\in S}\epsilon_i \geq \alpha_S\right] && (36)\\
&= 1 - \operatorname{erf}\left(\frac{\alpha_S}{\sqrt{2s}\sigma}\right) \to 0 && (37)
\end{aligned}$$

as $\sigma \to 0$. (Here we use the lemma 1.) Thus $\lim_{\sigma\to 0} p_S = 0$ for $S \neq N$ and

$$\lim_{\sigma\to 0}\varphi_i = \frac{(n-1)!}{n!} = \frac{1}{n}$$

since $p_N = 1$. $\|$

Proof of lemma 1: Define

$$R_\alpha \equiv \{(\epsilon_1,\ldots,\epsilon_n) \in \mathbb{R}^n | \epsilon_1 + \cdots + \epsilon_n \geq \alpha\}.$$

Then, we have

$$\mathbb{P}_n(\alpha) = \int_{R_\alpha} g(\epsilon_1,\ldots,\epsilon_n)d\epsilon_n \ldots d\epsilon_1,$$

where

$$g(\epsilon_1, \ldots, \epsilon_n) = f(\epsilon_1) \cdots f(\epsilon_n) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-(\epsilon_1^2 + \cdots + \epsilon_n^2)/2\sigma^2}. \tag{38}$$

Since equation (38) shows that the function g depends only on the radius from the origin, $r = \sqrt{\epsilon_1^2 + \cdots + \epsilon_n^2}$, the integral value does not change even if rotating the integration interval around the origin. Define a plane by

$$T_\alpha := \{(\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^n | \epsilon_1 + \cdots + \epsilon_n = \alpha\}.$$

By using the fact that the distance from the origin to the plane is $|\alpha|/\sqrt{n}$, we have

$$\begin{aligned} \mathbb{P}_n(\alpha) &= \int_{\alpha/\sqrt{n}}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-(\epsilon_1^2 + \cdots + \epsilon_n^2)/2\sigma^2} d\epsilon_n \cdots d\epsilon_1 \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{\alpha/\sqrt{n}}^{\infty} e^{-\epsilon_1^2/2\sigma^2} d\epsilon_1. \end{aligned}$$

Then, by using $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ and $\epsilon_1 = x/\sqrt{2}\sigma$, we can see that

$$\mathbb{P}_n(\alpha) = \frac{1}{2}\left(1 - \mathrm{erf}\left(\frac{\alpha}{\sqrt{2n}\sigma}\right)\right).$$

$\|$

Claim 1 (Convergence of the Normalized Shapley Value) *Suppose* $\sum_{i \in N} x_i = z$, $x_i \geq 0$ *for all* $1 \leq i \leq n$ *and* $\#\{i | x_i = 0\} = m < n$. *Then*

$$\lim_{\sigma \to 0} \phi_i(t = 1) = \begin{cases} \frac{1}{n-m} & \text{if } x_i > 0 \\ 0 & \text{if } x_i = 0 \end{cases} \tag{39}$$

which is same as the deterministic Shapley value and

$$\lim_{\sigma \to 0} \varphi_i(t = 1) = \begin{cases} \sum_{n-m-1 \leq s \leq n-1} \frac{m!s!}{(m-n+1+s)!n!} \left[1 - \frac{1}{\pi} \arccos\left(\sqrt{\frac{s+1}{n}}\right)\right] & \text{if } x_i > 0 \\ \frac{n-m}{m\pi} \sum_{n-m-1 \leq s \leq n-1} \frac{m!s!}{(m-n+1+s)!n!} \arccos\left(\sqrt{\frac{s+1}{n}}\right) & \text{if } x_i = 0. \end{cases} \tag{40}$$

Proof of Claim 1: First, we have

$$q_S \equiv \mathbb{P}\left[\sum_{i \in S} \epsilon_i \geq z - \sum_{i \in S} x_i\right] = \frac{1}{2}\left(1 - \mathrm{erf}\left(\frac{\alpha_S}{\sqrt{2s}\sigma}\right)\right)$$

by lemma 1. Then the normalized Shapley value ϕ_i of player i is

$$\begin{aligned}\phi_i &= 2\sum_{S\subset N\backslash\{i\}} \frac{s!(n-1-s)!}{n!}(q_{S\cup\{i\}} - q_S)\\ &= \sum_{S\subset N\backslash\{i\}} \frac{s!(n-1-s)!}{n!}\left[\operatorname{erf}\left(\frac{\alpha_S}{\sqrt{2s}\sigma}\right) - \operatorname{erf}\left(\frac{\alpha_{S\cup\{i\}}}{\sqrt{2(s+1)}\sigma}\right)\right].\end{aligned}$$

Without loss of generality, we will assume that $x_1 = \cdots = x_m = 0$ and $x_{m+1},\ldots,x_n > 0$. We have $\lim_{\sigma\to 0} q_S = 0$ if $\alpha_S > 0$ and $\lim_{\sigma\to 0} q_S = \frac{1}{2}$ if $\alpha_S = 0$, i.e. $[m{+}1,n] \equiv \{m{+}1,\ldots,n\} \subseteq S$. For $m+1 \le i \le n$, we have

$$\begin{aligned}\lim_{\sigma\to 0}\phi_i &= \sum_{[m+1,n]\backslash\{i\}\subset S\subset N\backslash\{i\}} \frac{s!(n-1-s)!}{n!} \qquad (\lim_{\sigma\to 0} q_S = 0)\\ &= \sum_{n-m-1\le s\le n-1} \binom{m}{n-1-s}\frac{s!(n-1-s)!}{n!}\\ &= \frac{m!}{n!}\sum_{n-m-1\le s\le n-1} \frac{s!}{(m-n+1+s)!}\\ &= \frac{m!(n-m-1)!}{n!}\sum_{n-m-1\le s\le n-1} \binom{s}{n-m-1}\\ &= \frac{m!(n-m-1)!}{n!}\sum_{n-m-1\le s\le n-1} \left[\binom{s+1}{n-m} - \binom{s}{n-m}\right]\\ &= \frac{m!(n-m-1)!}{n!}\frac{n!}{m!(n-m)!} = \frac{1}{(n-m)}\end{aligned}$$

and by efficiency and symmetry, we have $\lim_{\sigma\to 0}\phi_i = 0$ for $1 \le i \le m$.

In the case of stochastic Shapley value, we will use the definition of p_S again. Our aim is to prove the following equation : If $\alpha_S = z - \sum_{i\in S} x_i = 0$, then

$$p_S = 2\mathbb{P}\left[\sum_{i\in S}\epsilon_i \ge 0 \wedge \sum_{i\in N}\epsilon_i \ge 0\right] = 1 - \frac{1}{\pi}\arccos\left(\sqrt{\frac{s}{n}}\right).$$

We know that the area represented by

$$R_S \equiv \{(\epsilon_1,\ldots,\epsilon_n) \in \mathbb{R}^n | \sum_{i\in S}\epsilon_i \ge 0 \wedge \sum_{i\in N}\epsilon_i \ge 0\}$$

is an intersection of two half regions determined by two planes

$$\mu_S \equiv \{(\epsilon_1, \dots, \epsilon_n) \in \mathbb{R}^n | \sum_{i \in S} \epsilon_i = 0\}$$
$$\mu_N \equiv \{(\epsilon_1, \dots, \epsilon_n) \in \mathbb{R}^n | \sum_{i \in N} \epsilon_i = 0\}.$$

Since the probability distribution function

$$g(\epsilon_1, \dots, \epsilon_n) = f(\epsilon_1) \cdots f(\epsilon_n) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-(\epsilon_1^2 + \cdots + \epsilon_n^2)/2\sigma^2}$$

has spherical symmetry (i.e. only depends on $r = \sqrt{\epsilon_1^2 + \cdots + \epsilon_n^2}$), the value of

$$\mathbb{P}\left[\sum_{i \in S} \epsilon_i \geq 0 \wedge \sum_{i \in N} \epsilon_i \geq 0\right] = \int_{R_S} g(\epsilon_1, \dots, \epsilon_n) d\epsilon_1 d\epsilon_2 \cdots d\epsilon_n$$

only depends on the angle between two planes μ_S and μ_N. More precisely, let

$$v_S \equiv (a_1, a_2, \dots, a_n)$$
$$v_N \equiv (1, 1, \dots, 1)$$

where $a_i = 1$ if $i \in S$ and 0 otherwise. These vectors are normal vectors of μ_S, μ_N respectively, and the angle θ between two vectors are

$$\cos\theta = \frac{\langle v_S, v_N \rangle}{||v_S|| \cdot ||v_N||} = \frac{s}{\sqrt{sn}} = \sqrt{\frac{s}{n}}$$

for $S \neq \emptyset$. Then we have

$$p_S = 2\mathbb{P}\left[\sum_{i \in S} \epsilon_i \geq 0 \wedge \sum_{i \in N} \epsilon_i \geq 0\right] = 2\frac{\pi - \theta}{2\pi} = 1 - \frac{1}{\pi} \arccos\left(\sqrt{\frac{s}{n}}\right).$$

Now we will find $\lim_{\sigma \to 0} \varphi_i$. Without loss of generality, we will assume that $x_1 = \cdots = x_m = 0$ and $x_{m+1}, \dots, x_n > 0$. Then $\alpha_S = 0$ if and only if $[m+1, n] := \{m+1, \dots, n\} \subseteq S$.

For $m+1 \le i \le n$, by the previous observation, we have

$$\begin{aligned}
\lim_{\sigma\to 0}\varphi_i &= \sum_{[m+1,n]\setminus\{i\}\subseteq S\subseteq N\setminus\{i\}} \frac{s!(n-1-s)!}{n!} p_{S\cup\{i\}} \qquad (\lim_{\sigma\to 0} p_S = 0)\\
&= \sum_{[m+1,n]\setminus\{i\}\subseteq S\subseteq N\setminus\{i\}} \frac{s!(n-1-s)!}{n!}\left[1-\frac{1}{\pi}\arccos\left(\sqrt{\frac{s+1}{n}}\right)\right]\\
&= \sum_{n-m-1\le s\le n-1} \binom{m}{n-1-s}\frac{s!(n-1-s)!}{n!}\left[1-\frac{1}{\pi}\arccos\left(\sqrt{\frac{s+1}{n}}\right)\right]\\
&= \sum_{n-m-1\le s\le n-1} \frac{m!s!}{(m-n+1+s)!n!}\left[1-\frac{1}{\pi}\arccos\left(\sqrt{\frac{s+1}{n}}\right)\right].
\end{aligned}$$

Note that $\lim_{\sigma\to 0}\varphi_i$ is same for $m+1 \le i \le n$. Also, by symmetry, $\lim_{\sigma\to 0}\varphi_i$ is same for $1 \le i \le m$. Now put

$$\lim_{\sigma\to 0}\varphi_i = \begin{cases} a & \text{if } 1 \le i \le m\\ b & \text{if } m+1 \le i \le n. \end{cases}$$

By efficiency, we have $ma + (n-m)b = 1$, so

$$a = \frac{1}{m}\left(1-(n-m)\sum_{n-m-1\le s\le n-1} \frac{m!s!}{(m-n+1+s)!n!}\left[1-\frac{1}{\pi}\arccos\left(\sqrt{\frac{s+1}{n}}\right)\right]\right).$$

By the way, we have

$$\sum_{n-m-1\le s\le n-1} \frac{m!s!}{(m-n+1+s)!n!} = \sum_{n-m-1\le s\le n-1} \binom{m}{n-1-s}\frac{s!(n-1-s)!}{n!} = \frac{1}{n-m}$$

so the above equation is same as

$$a = \frac{n-m}{m\pi}\sum_{n-m-1\le s\le n-1} \frac{m!s!}{(m-n+1+s)!n!}\arccos\left(\sqrt{\frac{s+1}{n}}\right).$$

‖

A numerical Example for $\Delta^N > 1$*:* Take $\sigma = 1$, $x_1 = 4, x_2 = 0, z = 2$ so that $x_1 + x_2 > z$.

We have

$$\begin{aligned}
\mathbb{P}[\epsilon_1 + \epsilon_2 \geq z - (x_1 + x_2)] &= \frac{1}{2}\left(1 - \text{erf}\left(\frac{z - (x_1 + x_2)}{2\sigma}\right)\right) \\
&= \frac{1}{2}(1 - \text{erf}(-1)) \\
&= 0.92135... \\
\mathbb{P}[z - x_1 \leq \epsilon \leq z - x_2] &= \frac{1}{2}\left(\text{erf}\left(\frac{z - x_2}{\sqrt{2}\sigma}\right) - \text{erf}\left(\frac{z - x_1}{\sqrt{2}\sigma}\right)\right) \\
&= \frac{1}{2}\left(\text{erf}(2\sqrt{2}) - \text{erf}(-\sqrt{2}))\right) \\
&= 0.97721...
\end{aligned}$$

Therefore, it leads to

$$\Delta^N = \frac{\mathbb{P}[z - x_1 \leq \epsilon \leq z - x_2]}{\mathbb{P}[\epsilon_1 + \epsilon_2 \geq z - (x_1 + x_2)]} = 1.06063... > 1.$$

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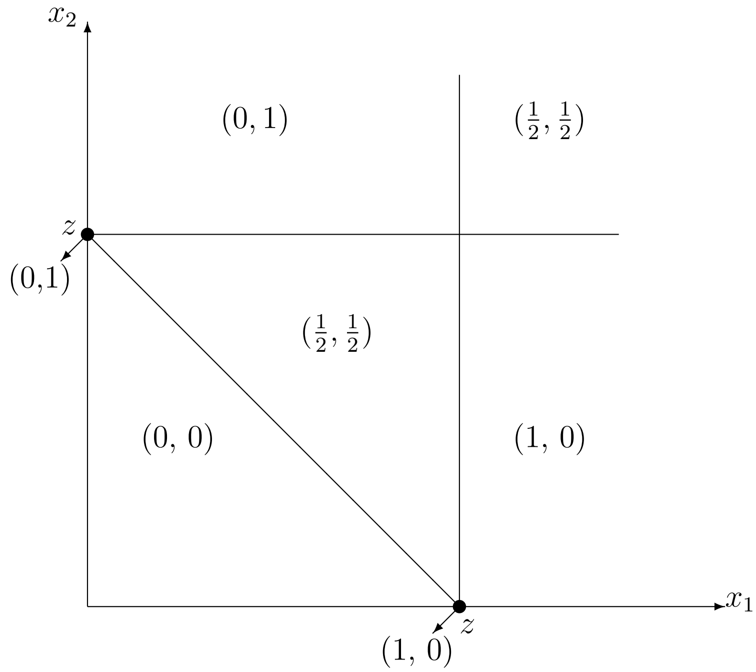

Figure 1. (φ_1, φ_2) in the deterministic case