A Theory of Consumer Referral

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Abstract

In this paper, we consider the network as an alternative communication channel to the undirected advertising in the market. The main distinctive feature of the network-embedded transaction is the dependency of buyers' purchasing behavior, which makes all consumers not equally valuable to a seller. We characterize the optimal behavior of a seller and consumers in a network. A seller's strategy of paying referral fees can be understood as a way of price discrimination between more valuable consumers and less valuable ones. Numerical simulations demonstrate that the social network may be either overutilized (if the referral cost is high) or underutilized (if the referral cost is low).

Keywords: Consumer Referral, Network, Price Discrimination

JEL Classification Codes: D40, L10

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1 Introduction

A trade between a seller and a buyer can occur only after any form of communication informs them of each other's presence. To promote products, sellers often use advertisements in the market, which is considered as one of the most typical ways of communication. Usually, advertising requires a huge amount of expenditures and it is by nature *undirected* in the sense that it targets numberless anonymous consumers. It might be regarded as a quick and efficient way of information propagation notwithstanding its high initial cost.

Recently, we observe that an alternative way of marketing using the human network, so-called network marketing (or multi-level marketing or referral marketing) is replacing the traditional advertising.[1] Many companies, like Mary Kay, Amway, Shaklee, Electrolux, Tupperware and Herbalife rely heavily on this form of marketing. Aside from them, abundant examples can be found. To name a few, one may have noticed the advertisement of Burger King for job applicants, "If you bring a friend, you will get a bonus." An apartment resident who successfully refers a new resident gets paid a referral fee. Some sports clubs allow a discount on the membership fee for a member who brings another. Overall, it is a recent trend that the volume of 'network-embedded' transactions is growing disproportionately, possibly due to the development of the Internet which enables consumers to lower the referral cost. Casual observations show network transactions are used particularly in situations where the firm targets only the geographically limited consumers, or targeted consumers interact frequently enough to be informed of each other's needs, as in examples of apartment rental, skin care service, plastic surgery for actresses, or overseas accident insurances for foreign visiting scholars.

The biggest advantage of network marketing is to utilize the existing social network to avoid the huge amount of investment costs in advertising. Firms provide various incentive schemes to induce consumers to recommend their products to their neighbors. Since social networks are built from years of relationships and exchanges between individuals, they can work as a channel of information flows between the members who trust each other.

One of the main features that distinguish the network transaction from the traditional

[1] According to the Direct Selling Association (1999), annual sales of multi-level marketing firms grew from 13 billion to nearly 23 billion dollars between 1991 and 1998.

transaction in the market is the dependency of consumers' purchasing behavior. Since most transactions in a network are made sequentially, all consumers are not equally valuable to a seller. Offering a referral fee is an effective way of price discrimination among consumers in network transactions.

We set up the model of network transactions as follows. On the demand side, a finite group of potential consumers are arrayed on a line. They are linked with one another in such a way that each consumer has exactly one neighbor whom he can refer. A consumer is unaware of the availability of the product unless he gets referred. Once a consumer gets referred, he may buy the product and refer his neighbor. On the supply side, there is a monopolist which prepares for selling to the group of consumers and is linked with only one consumer. The monopolist offers a referral fee when a consumer successfully refers his neighbor.

We summarize main results of the paper as follows. The last consumer of the network buys if his valuation is greater than the price. The penultimate consumer is more likely to buy than the last consumer, considering the possibility of earning a referral fee when he can successfully refer the last consumer. By the induction method, we show that a consumer who is earlier in the chain of transactions is more likely to buy. This result implies that a monopolist can effectively discriminate consumers based on their position on the network, since the likelihood of successful referral differs across consumers' positions. The monotonicity of the likelihood of buy-and-referral also implies that, if the penultimate consumer buys and refers, all other consumers who buy must refer as well. Therefore we have two extreme equilibria, depending on the size of referral costs. For a small referral cost, in which case the monopolist will find it profitable to encourage consumer referrals, all consumers who buy will refer, while none of the consumers will refer for a large referral cost.

We, then, compare the monopolist's profit and social efficiency of the network transaction and the traditional market transaction. Network transactions have the potential of improving the social welfare since it utilizes the already-built social network for the diffusion of its product. On the other hand, a sequential manner of buy-and-referral and the consequent interdependency of consumers' behavior can be a source of inefficiency. We show that social networks can be both over- and under-utilized.

In Section 6, we take the local nature of the social network more seriously, and assume

that each consumer observes his neighbor's valuation. We show that the availability of the local information improves both the monopolist's profits and social welfare. This is because the social wastes of unsuccessful referral efforts are reduced, if a consumer uses local information to screen his neighbor on the basis of the observed valuation.

In Section 7, we introduce more realistic features in the valuations of consumers by assuming that they are correlated. If a product is of high value to one consumer, it is likely that the product will have high value to all other consumers. Then, we demonstrate that a consumer's referral decision after making a purchase depends on his own valuation. This is the direct consequence of correlated valuations that enable only a consumer with a high valuation to believe that his neighbor will be highly likely to buy when he refers.

2 Model

To model network-embedded transactions, we begin with some useful concepts in graph theory. Suppose we have a population of n individuals, denoted by N, such that each individual $i \in N$ has a set of social "neighbors" $N_i \subset N \setminus \{i\}$ with whom he is linked or connected. Each individual is a neighbor to his neighbors, so that $j \in N_i$ if and only if $i \in N_j$. If i and j are neighbors, they are said to be connected or linked. The members of the population can be represented as vertices of an undirected graph g, with an edge connecting every pair of neighbors i and j.[2] We denote $g_{ij} = 1$ if and only if i and j are neighbors to each other, and zero otherwise. Individuals i and j can be also connected indirectly if there is a sequence of vertices $i, i', i'', ..., j$ such that each member of the sequence is connected to his neighbor. Such a sequence is called a *path* between i and j. Let d_i denote the cardinality of N_i, that is, the number of individuals with whom i is linked with. The *degree* of the graph $d(g) = \sum_i d_i/n$ is the average number of neighbors to whom an individual is connected. The graph is *regular* if $d_i = d_j$ for all $i, j \in N$, in which case all individuals have the same number of neighbors. The regular graph of degree $n-1$ is called *complete* graph.

The basic model of consumers' network is constructed as follows. There is only one seller who prepares for launching a product to a group of consumers in a community. Let

[2] Allowing more than one edge is a waste since an edge linking a pair of individuals only represents the fact that individuals are neighbors to each other.

$N = \{1, 2, ..., n\}$ denote the group of consumers. The web of relationships among consumers is represented by a path graph[3] such that $g_{ij} = 1$ for all $j = i - 1$ for all $i \geq 2$ and zero otherwise. From consumer i's point of view, consumers $j > i$ are called *predecessors* and consumers $j < i$ are called *successors*. We further assume that the monopolist does not have a link with anyone else in the community except consumer n. Hence the extended path graph now consists of $n + 1$ players adding the monopolist, called player 0.

The monopolist can produce a unit at a constant marginal cost, $c > 0$. Consumer i's valuation for the product is a random variable v_i, which is i.i.d. over the range of $[\underline{v}, \bar{v}]$ according to the continuous density function $f(.)$ and corresponding distribution function $F(.)$. A consumer does not know the availability of the product, unless he is either referred or directly connected to the monopolist. Each consumer buys at most one unit of the product at the price of p. Only consumers who buy one may refer their neighbor (except consumer 1). Making a referral costs the referee $\delta > 0$.[4] If a consumer successfully refers his neighbor and so his neighbor buys one, he will be awarded a referral fee.[5] For the most part of the paper, we will exclude the possibility that a consumer who does not buy himself refer his neighbor, because his recommendation would not be as credible as the recommendation by one who buys himself.[6] However, we will discuss the alternative case as well in which even a consumer who does not buy is entitled to refer.

The game proceeds as follows. The monopolist decides the price p and the referral fee r that apply to all consumers. If consumer n does not buy, the game ends. If he buys, he may refer the product to consumer $n - 1$. Once referred, consumer $n - 1$ faces the identical decision of buy-and-referral. This chain of buy-and-referral continues until no consumer is left to buy. Finally, it is assumed that the structure of the social network, the game, and all

[3] A graph consisting of a single path between any pair of individuals is called a path graph.

[4] The referral costs, for example, include monetary costs such as making phone calls, as well as non-monetary costs such as efforts to convince neighbors to buy the product.

[5] It is disputable who should get paid a referral fee. For example, when consumer i buys, it may be rational to pay all his predecessors $j > i$. In fact, this is a usual strategy employed by network marketing firms. By paying (potentially different) referral fees to all (or some) predecessors, the monopolist can provide consumers an incentive to discriminate their neighbors on the basis of the number of their successors. Therefore a paying-all-predecessors scheme has a potential to improve the monopolist's profit.

[6] Moreover, in many practices of network marketing, only consumers who buy themselves are allowed to refer other potential consumers.

previous moves of other players are common knowledge.[7]

3 Traditional Market Transaction

The traditional market can be defined as a mechanism by which sellers (or a seller) and buyers interact for transactions. By contrast, a network is a group of sellers and buyers who are interlinked. We will call the transaction in the former trading environment simply "market transaction" and the transaction in the latter environment "network transaction".

Suppose $g_{0i} = 1$ for all $i \in N$, in other words, the monopolist is linked with all consumers in the community. This corresponds to the market formation.

Consumer i will buy one if and only if his valuation is greater than the price, $v_i \geq p$. Hence the market demand will be $n(1 - F(p))$. The monopolist will choose the optimal price p_M^* to solve

$$\max_p \pi_M \equiv (p - c)\, n\,(1 - F(p)).$$

The first order condition implies $p_M^* = c + (1 - F(p_M^*))/f(p_M^*)$. Let π_M^* denote the maximized profit in the market transaction.

4 Network Transaction

In this section, we characterize the outcome of network-embedded transactions. First, we will discuss the consumers' behavior of buy-and-referral. Then we proceed to discuss how the monopolist chooses the price and the referral fee to maximize profits, given the behavioral rule of consumers.

4.1 Strategies of Consumers

In network-embedded transactions, each consumer's choice of buy-and-referral depends on the behavior of other consumers in two principle ways. First, by the assumption, a consumer

[7] Although consumers $i \neq n$ know the structure of the network, they do not know how to reach the monopolist (communicate with him), until they are referred, as far as they have no direct links with the monopolist.

can only get information about the product if and only if all of his predecessors buy and refer. Second, his choice of buy-and-referral depends on the expected behavior of all his successors.

A consumer's choice can be represented in a recursive manner as follows: consumer i buys if and only if

$$v_i \geq \tilde{v}_i \equiv p - \max\left\{r\left(1 - F\left(\tilde{v}_{i-1}\right)\right) - \delta, 0\right\}, \forall i \geq 2, \text{ and } v_1 \geq p \tag{1}$$

where $\tilde{v}_i$ denotes the minimum valuation for consumer i to buy. It can be explained as follows. If consumer i buys one for $i \geq 2$, he derives the valuation at the price of p, but he can earn r at the referral cost of δ with probability $1 - F\left(\tilde{v}_{i-1}\right)$ when he refers consumer $i-1$, implying that he refers only when $r\left(1 - F\left(\tilde{v}_{i-1}\right)\right) > \delta$.[8] Since consumer 1 has no successor to refer, all he has to care about is his net valuation from his own purchase. Notice that a high r and a low p are substitutable tools for inducing consumer referrals. The above recursive relationship implies the following result.

Proposition 1 *Suppose that $\tilde{v}_k = p$ and $\tilde{v}_{k+1} < p$ for some k. Then the minimum valuation decreases as the consumer is located earlier in the chain of network:*

$$\tilde{v}_n < \tilde{v}_{n-1} < \tilde{v}_{n-2} < ... < \tilde{v}_k = ... = \tilde{v}_1 = p.$$

In particular, if $k = 1$, $\tilde{v}_n < \tilde{v}_{n-1} < \tilde{v}_{n-2} < ... < \tilde{v}_1$.

Proof. See the appendix.

Note that equation (1) implies that if $\tilde{v}_i < p$, $r\left(1 - F\left(\tilde{v}_{i-1}\right)\right) - \delta > 0$, which in turn implies that consumer i who buys one will refer his neighbor. Equation (1) also implies that if $\tilde{v}_i = p$, consumer i will not refer even when he buys one himself, since the expected gain from referral is not sufficient to cover his referral cost, i.e., $\delta \geq r\left(1 - F\left(\tilde{v}_{i-1}\right)\right)$. In particular, since $\tilde{v}_1 = p$, the recursive relationship in (1) implies the following:

$$\tilde{v}_2 < p \Longleftrightarrow r\left(1 - F\left(p\right)\right) > \delta. \tag{2}$$

[8] We implicitly assume that the degenerate case of indifference is resolved against referral.

The relationship in (2) says that if consumer 2 buys one, he also refers. Since consumer 2 is least likely to buy (except consumer 1) by Proposition 1, equation (2) implies that all consumers who buy one will always refer. Therefore we term the inequality of $r(1-F(p)) > \delta$ as the referral condition, or abbreviated as RC, in the sense that, if this condition is satisfied, all consumers who buy will also refer. This is formally stated as follows.

Proposition 2 $[RC] \Longleftrightarrow \tilde{v}_n < \tilde{v}_{n-1} < \tilde{v}_{n-2} < ... < \tilde{v}_3 < p$.

Proof. See the appendix.

Proposition 2 implies that, if [RC] is satisfied, all consumers who buy one will always refer and, if not satisfied, no consumer will refer. This is formally shown as follows.

Theorem 1 *Either $k = 1$ or $k = n$ where k is the integer defined in Proposition 1.*

Theorem 1 implies that only two outcomes are feasible in equilibrium: either consumer n does not refer ($k = n$: *no referral equilibrium*), or if he refers, all the subsequent consumers who buy refer their neighbor ($k = 1$: *all referral equilibrium*). Hence it cannot occur in equilibrium that a consumer, except consumer n, does not refer his neighbor after he buys one himself.

4.2 Strategies of the Monopolist

Given the consumers' behavior described above, the monopolist will choose the optimal price p and referral fee r to maximize the expected profits in the network-embedded transaction. By Theorem 1, if p and r are such that [RC] is met, the profit is

$$\begin{aligned} \pi_N(n,\delta) &\equiv (p-c)P(n,1) + (p-c-r)\sum_{k=2}^{n} P(n,k) \\ &= \sum_{k=1}^{n} [k(p-c) - (k-1)r] P(n,k) F(\tilde{v}_{n-k}), \end{aligned}$$

where

$$P(n,k) \equiv \prod_{i=n-k+1}^{n} [1 - F(\tilde{v}_i)], \quad F(\tilde{v}_0) \equiv 1 \tag{3}$$

and if [RC] is not met, $\pi_0 \equiv (p-c)(1-F(p))$. Note that $k(p-c) - (k-1)r$ is the monopolist's profit when exactly k consumers buy, and the term $P(n,k)F(\tilde{v}_{n-k})$ represents

the probability that this will happen. We interpret $P(n,k)\,F(\tilde{v}_{n-k})$ as the probabilistic demand.

Let $\pi_N^*(n,\delta)$ and π_0^* denote the maximized values of $\pi_N(n,\delta)$ and π_0, respectively. Let p_N^* and r_N^* denote the optimal price and referral fee when [RC] is met. Note that the optimal price in the case where [RC] is not met is identical to the optimal price under the market transaction, p_M^*.

Now we show that the monopolist prefers the *all referral equilibrium* for a small value of δ, and the *no referral equilibrium* for a large value of δ. The intuition is as follows. If the referral cost is high, the monopolist must compensate consumers either by paying a high referral fee or by charging a low price. This will reduce the profit. Therefore if the referral cost is high enough, the monopolist will forgo the network of consumers and sell only to consumer n. This is formally shown in Theorem 2.

Theorem 2 *For any n, there exists $\underline{\delta}(n)$ such that the monopolist prefers the all referral equilibrium for all $\delta < \underline{\delta}(n)$ and prefers the no referral equilibrium for all $\delta \geq \underline{\delta}(n)$.*[9]

Proof. See the appendix.

One can understand the monopolist's strategy of paying referral fees as a way of price discrimination. As an inducement to more demand through referrals, the monopolist gives a discount to consumers except the last one who has no other to refer. One interesting feature in this model is that, although the monopolist discriminates (nominal) prices only between a consumer with a successor and a consumer without a successor, the effect of the price discrimination differs across all consumers. This is because a consumer's benefit from making a referral is proportional to the probability that his neighbor buys and this probability is decreasing in the chain of referrals. If we interpret $\tilde{v}_i$ as the generalized price charged to consumer i in the sense that it is the actual cost that he must bear for a purchase, the

[9] If δ is small enough such that $\delta < (p_M^* - c)(1 - F(p_M^*)) = \pi_M^*$, the monopolist can earn a higher profit in the *all referral equilibrium* than in the *no referral equilibrium*, by choosing (p_M^*, r') where $r' \in (\delta/(1 - F(p_M^*)), p_M^* - c)$. This is because the monopolist charges the same monopoly price to the first consumer and can also induce him to refer the next consumer. Note that the above condition for δ implies that $\underline{\delta}(n)$ is actually large enough that the monopolist will prefer the *all referral equilibrium* for a wide range of parameters.

monopolist charges different generalized prices to all consumers. Specifically, the generalized price for the first consumer is lowest since he is most valuable to the monopolist, while that for the last consumer is highest.

One crucial feature that distinguishes the network transaction from the market transaction is that later consumers' purchasing behavior is dependent on earlier consumers' decisions. So, a later consumer's high valuation may not be realized no matter how high it may be, unless his predecessors buy one. Considering this, the monopolist gives discounts to earlier consumers. Then, earlier consumers, who would not buy in the market situation, may buy one. Therefore, the volume of sales through a network can possibly be larger than that in the market transaction. However, despite the price discrimination, it is not possible that the profit from the network transaction exceeds the profit from the market transaction, since large sales through a network could be achieved only by paying high referral fees.

Theorem 3 $\pi_N^* < \pi_M^*$.

Proof. See the appendix.

Finally, we investigate how the monopolist tunes its price and referral fee to a change in underlying situations. Given the complicated form of the profit function, we are not able to draw closed-form solutions for the optimal control variables. Hence we rely on numerical simulations for comparative statics. For the purpose, we assume that $F(.)$ is uniformly distributed over $[0, 1]$.

The overall lesson from simulation results is that monotonicity of the optimal price and referral fee with respect to a change in underlying parameters should not be taken for granted. Figure 1 shows nonmonotonicity of the control variables as the population size changes. The intuition is as follows. As Proposition 1 suggests, each consumer is, in general, more likely to buy as there are more consumers. Facing this, the monopolist has two conflicting incentives. One is to exploit this situation by raising the price and paying a lower referral fee in order to potentially generate a higher profit. The other is to encourage consumer referrals further by lowering the price and paying a higher referral fee in expectation of generating even higher demand. One incentive dominates the other, depending on a change in the demand resulting from the increased number of consumers. Therefore the overall effect is ambiguous.

Figure 2 illustrates the case where the price responds nonmonotonically to changes in the marginal cost. This is surprising, since normally it is expected that a higher marginal cost will lead to a higher price. In the network transaction, however, the argument must be completed by incorporating the adjustment in the referral fee. A higher marginal cost may increase the price but lower the referral fee, because an increase in the referral fee has a smaller effect of enhancing the profit through an increase in demand when associated with a higher marginal cost. Since a lower referral fee implies a lower cost for the monopolist, it may result in a fall in the price. Figure 3 examples the situation where the price changes nonmonotonically in response to changes in the referral cost δ.

4.3 Discussion

Buy-to-Refer Constraint

Suppose a consumer is not required to buy himself for referring his neighbor. In this case, two decisions of consumer i, buying decision and referral decision, are independent, and consumers' purchasing decisions are no longer interdependent. Specifically, consumer i is concerned only about v_i in his buying decision, whereas he is concerned about v_{i-1} in his referral decision. It follows clearly that consumer i buys one if and only if $v_i \geq p$ and that he refers if and only if [RC] condition is satisfied. The monopolist's profit is then $\pi_0 + (p - c - r)(n - 1)(1 - F(p))$ for all p and r satisfying [RC] condition and π_0 if [RC] condition is not satisfied.

Figure 4 compares the profits with and without the buy-to-refer constraint in the case that $n = 4$. Region B is the area in which the profit with the constraint exceeds the profit without it, while region A is the area in which the profit without the constraint is greater. The monopolist's buy-to-refer policy can have an advantage over the other. By requiring consumers to buy for referring their neighbor, the monopolist can induce earlier consumers to buy with higher probability, whereas later consumers would be induced to be more likely to buy without the constraint, due to their extended opportunities of getting referred. As the referral cost is higher so that the referral fee is set to be higher, it becomes more costly for the monopolist to pay referral fees, so it will find it in his interest to impose the buy-to-refer constraint.

Uncertainty about the Population Size

We will discuss the situation in which the monopolist is uncertain of the population size, while consumers are fully informed of the network structure.

We assume that the number of consumers, n, follows a geometric distribution whose density function is given by $h(n) = \beta(1-\beta)^{n-1}$, where $\beta \in (0,1)$ and $n = 1, 2, \cdots$. The interpretation of this distribution is that (i) the monopolist is not sure about the existence of other consumers than the first consumer with whom he is linked, and that (ii) the probability that $(i+1)$-th consumer exists given that i-th consumer exists is $1-\beta$. In this case, n is usually called a waiting time (until this sequence of consumers stops).

Since consumers know the network structure and the sequence of consumers stop in a finite time with probability one, consumers' decisions remain the same. The monopolist can calculate the expected profit based on his computation of the probability that there are i consumers, $i = 1, 2, \cdots$. Then, he can choose the optimal price and the referral fee to maximize the expected profit. The dependency of a consumer's purchasing decision on his location remains unaffected by the assumption that the population size is uncertain to the monopolist.

5 Market vs. Network

In this section, we consider the monopolist's choice between the market and the network transaction and discuss the social efficiency.

Selling products in a market requires marketing which involves a considerable amount of investment costs in advertising. Let the monopolist's advertising costs be A. We can interpret A as the cost of forming links to all consumers in the population. First, we compare profits in the market and the network-embedded transaction. The monopolist will prefer to market through the existing social network if and only if $\pi_N^* > \pi_M^* - A$. Notice that π_N^* depends on δ and is nonincreasing in δ.

Next, let us compare social welfares under two modes of transaction. The social welfare associated with the market transaction is

$$SW_M = \sum_{k=1}^{n} \int_p^{\bar{v}} (v_k - c) f(v_k | v_k \geq p) \, dv_k \, [1 - F(p)] - A$$

$$
\begin{aligned}
&= n\left[1-F\left(p\right)\right]\left[\int_p^{\bar{v}}\left(p-c\right)f\left(v|v\geq p\right)dv+\int_p^{\bar{v}}\left(v-p\right)f\left(v|v\geq p\right)dv\right]-A \\
&= n\left[1-F\left(p\right)\right]\left(p-c\right)+n\left[1-F\left(p\right)\right]\int_p^{\bar{v}}\left(v-p\right)f\left(v|v\geq p\right)dv-A \\
&= \left(\pi_M-A\right)+CS_M,
\end{aligned}
$$

where CS_M denotes the consumer surplus in the market transaction:

$$
CS_M = n\left[1-F\left(p\right)\right]\int_p^{\bar{v}}\left(v-p\right)f\left(v|v\geq p\right)dv.
$$

The social welfare associated with the network transaction is

$$
\begin{aligned}
SW_N &= \sum_{k=1}^{n}\left[\int_{v_{n-k+1}}^{\bar{v}}\left(v_{n-k+1}-c\right)f\left(v_{n-k+1}|v_{n-k+1}\geq\tilde{v}_{n-k+1}\right)dv_{n-k+1}-\delta\right]P\left(n,k\right) \\
&= \pi_N+CS_N,
\end{aligned}
$$

where CS_N denotes the consumer surplus in the network transaction:

$$
CS_N = \sum_{k=1}^{n}\left[\begin{array}{c}\int_{v_{n-k+1}}^{\bar{v}}\left(v_{n-k+1}-p\right)f\left(v_{n-k+1}|v_{n-k+1}\geq\tilde{v}_{n-k+1}\right)dv_{n-k+1} \\ +r\cdot I_{\{k\neq 1\}}-\delta\end{array}\right]P\left(n,k\right)
$$

where I_ϕ is equal to one if ϕ holds, and zero otherwise.

The first best outcome is that all consumers with $v_i \geq c$ buy. In the market transaction, a consumer buys if and only if $v_i \geq p_M^*$. Since $p_M^* > c$, there occur some efficiency losses. On the other hand, the network transaction may involve efficiency losses as well, since consumers with $v_i \geq c$ may not get referred when the chain of referrals discontinues.

Figure 5 and 6 illustrate four regions of profits and social welfares in the (A, δ)-dimension for $c = 0.01$ under the uniform distribution. Region A represents the case where $\pi_M - A \geq \pi_N$ and $SW_M \geq SW_N$. Region D is the case where the network transaction is desirable socially as well as to the monopolist, that is, $\pi_M - A < \pi_N$ and $SW_M < SW_N$. Clearly, if the advertising cost is high and the referral cost is low, the monopolist chooses to trade through the network, which is socially efficient. Region B and C represent the region of under- and over-utilization of social networks, respectively. In other words, $\pi_M - A \geq \pi_N$ and $SW_M < SW_N$ in region B and $\pi_M - A < \pi_N$ and $SW_M \geq SW_N$ in region C. The intuition for both possibilities goes as follows. Although consumers apparently bear the referral cost, they are remunerated enough for it by receiving referral fees, because they would not refer

if the referral fees were not enough. This means that it is the seller who actually bears the referral cost. If δ is high, the seller must pay a high referral fee, thereby pushing up a price. Hence, for high δ, social welfare associated with the network transaction can be smaller than that associated with the market transaction, while the profit in the network transaction is still larger than the profit in the market transaction, and vice versa for low δ.

6 Network of Informed Buyers

In a social network, the strength of relationships varies across pairs of individuals. Closely-linked members share a great deal of information and know each other's relevant characteristics, while distant pairs of members do not have much to share. In this section, we assume that each consumer observes his neighbor's valuation (but not the valuation of any other consumers). We maintain the assumption that the monopolist only knows the distribution of consumers' valuations.

6.1 Strategies of Consumers

Consumer 1's decision remains unaffected since he has no neighbor to refer. He will continue to buy if and only if $v_1 \geq p$. Consider the decision of consumer 2. Since he is informed of v_1, he knows whether $v_1 \geq p$ or $v_1 < p$, in other words, whether consumer 1 will buy or not if referred. His minimum valuation for a purchase depends on his expectation about consumer 1's purchase. If he observes that $v_1 \geq p$, he will buy and refer if

$$v_2 \geq \tilde{v}_2(1) \equiv p - r + \delta.$$

On the other hand, if he observes that $v_1 < p$, he will buy and not refer if and only if $v_2 \geq \tilde{v}_2(0) \equiv p$. Hence the behavioral rule of consumer 2 can be summarized as follows:

$$\text{Consumer 2 will} \begin{cases} \text{buy regardless of } v_1 & \text{if } v_2 \geq p \\ \text{buy as long as } v_1 \geq p & \text{if } v_2 \in [p - r + \delta, p) \\ \text{never buy} & \text{if } v_2 < p - r + \delta \end{cases} \tag{4}$$

Now consider the decision of consumer 3. He observes v_2, but not v_1. This, in turn, generates uncertainty over the behavior of consumer 2. Similar to consumer 2, consumer 3

will buy regardless of v_2 if $v_3 \geq p$, and will never buy if $v_3 < p - r + \delta$. For $p - r + \delta \leq v_3 < p$, there are three possibilities. If $v_2 \geq p$, consumer 3 knows that consumer 2 will buy if he refers. Hence consumer 3 will buy and refer. If $v_2 < p - r + \delta$, consumer 3 is sure that his neighbor will not buy, and thus he will not buy either. If $p - r + \delta \leq v_2 < p$, consumer 3 is not sure whether consumer 2 will buy or not. In this case, however, he can compute the probability that consumer 2 will buy, which, according to (4), corresponds to the probability that consumer 1 buys. This probability is $\alpha \equiv \Pr(v_1 \geq p) = 1 - F(p)$. Reasoning in this way, consumer 3 knows that consumer 2 will buy if and only if consumer 1 will buy, and this leads him to buy if

$$v_3 \geq \tilde{v}_3(\hat{\alpha}_2) \equiv p - \max\{r\hat{\alpha}_2 - \delta, 0\},$$

where $\hat{\alpha}_2$ is the *ex post* probability that consumer 2 will buy:[10]

$$\hat{\alpha}_2 = \begin{cases} 1 & \text{if } v_2 \geq p \\ \alpha & \text{if } v_2 \in [p - r + \delta, p) \\ 0 & \text{if } v_2 < p - r + \delta \end{cases}$$

Although the behavioral rule of consumer 3 gives a useful guideline for understanding it, the behavioral rule of the predecessors is much more complicated. A consumer $i \geq 4$ who observes $p - r + \delta \leq v_{i-1} < p$ has to compute the probability that consumer $i - 1$ buys, which depends on all possible realized valuations of successors. In a series of propositions, we attempt to give a characterization of consumers' behavior.

Proposition 3 *If consumer i who has valuation v_i buys in equilibrium, so does consumer i who has valuation $v_i' > v_i$.*

Proof. See the appendix.

Proposition 3 implies that there exists a minimum valuation for consumer i to buy, $\tilde{v}_i$, conditional on his observation on v_{i-1}. In other words, consumer i buys one if and only if $v_i \geq \tilde{v}_i$. Specifically, the minimum valuation of consumer i can be defined as follows.

$$\tilde{v}_i = p - \max\{\hat{\alpha}_{i-1} r - \delta, 0\},$$

[10] This is *ex post* in the sense that consumer 3 is informed of v_2.

where $\hat{\alpha}_{i-1}$ is the *ex post* probability that consumer $i-1$ will buy.

Part of consumer 3's behavioral rule can be easily generalized to the behavior of all other consumers as follows.

Proposition 4 *Consumer i buys if $v_i \geq p$ and never buys if $v_i < p - r + \delta$ regardless of v_{i-1}, for all $i \geq 2$.*

Proof. Trivial.

Notice that consumer i who has valuation $v_i \geq p$ buys but does not refer, if $v_{i-1} < p-r+\delta$ (implying $\hat{\alpha}_{i-1} = 0$). He will refer only when $\hat{\alpha}_{i-1} r > \delta$. However, if $p - r + \delta \leq v_i < p$, consumer i who buys must refer, otherwise he will make a loss from the purchase, as summarized in Proposition 5.

Proposition 5 *Suppose $p - r + \delta \leq v_i < p$. Then, consumer i refers if he buys, and he buys if and only if $v_i \geq p - \hat{\alpha}_{i-1} r + \delta$.*

Proof. See the appendix.

The next proposition gives the formula for $\hat{\alpha}_{i-1}(v_{i-1})$. Integration in the proposition comes from the fact that consumer i can observe v_{i-1} but cannot observe v_{i-2}.

Proposition 6 *For $i \geq 3$,*

$$\hat{\alpha}_{i-1}\left(v_{i-1}\right) = \int_{\underline{v}}^{\overline{v}} I_{\{v_{i-1} \geq \tilde{v}_{i-1}(\hat{\alpha}_{i-2})\}} dF\left(v_{i-2}\right),$$

and $\hat{\alpha}_1 = 1$ if $v_1 \geq p$ and zero otherwise, where

$$\tilde{v}_{i-1} = \begin{cases} p - \hat{\alpha}_{i-2}\left(v_{i-2}\right) r + \delta & \text{if } v_{i-2} \geq p - r + \delta \\ p & \text{if } v_{i-2} < p - r + \delta \end{cases} \quad \text{for } i \geq 4,$$

and $\tilde{v}_2 = p - r + \delta$ if $v_1 \geq p$ and p otherwise.

Proof. This follows directly from definitions of $\hat{\alpha}_{i-1}$ and $\tilde{v}_{i-1}$.

From Proposition 4, it is clear that $\hat{\alpha}_{i-1} = 1$ if $v_{i-1} \geq p$ and $\hat{\alpha}_{i-1} = 0$ if $v_{i-1} < p - r + \delta$. Also, it is easy to see that if $p - r + \delta \leq v_2 < p$,

$$\hat{\alpha}_2 = \int_{\underline{v}}^{\overline{v}} I_{\{v_2 \geq \tilde{v}_2\}} dF\left(v_1\right) = \int_{\underline{v}}^{p} I_{\{v_2 \geq p\}} dF\left(v_1\right) + \int_{p}^{\overline{v}} I_{\{v_2 \geq p-r+\delta\}} dF\left(v_1\right) = \alpha.$$

Proposition 7 *(i)* $\hat{\alpha}_i(v)$ *is nondecreasing in* i *for given* v*. (ii)* $\tilde{v}_i(\hat{\alpha}_{i-1}(v))$ *is nonincreasing in* i *for given* v*.*

Proof. See the appendix.

This proposition says that, given the same observation of the subsequent consumer's valuation, an earlier consumer is more likely to buy one. This is a direct counterpart of Proposition 1.

The crucial difference in their behavior between uninformed and informed consumers is that an informed consumer who buys may not refer even when [RC] is satisfied. A consumer who buys will not refer if he knows that the next consumer does not need the product very much. If he knows that the next consumer is eager to consume one, he will refer. If the next consumer's need is not so strong, he will refer only when he believes that the second next consumer will need it eagerly.

For the analysis below, we will restrict our attention to the case where [RC] is satisfied.[11] Then, we have the following result.

Proposition 8 *Under [RC] condition,* $\hat{\alpha}_{i-1}(v_{i-1})\, r > \delta$ *for all* $v_{i-1} \in [p - r + \delta, p)$*.*

Proof. See the appendix.

As Proposition 8 suggests, imposing [RC] condition is to avoid the trivial case where consumer i observing $v_{i-1} \in [p - r + \delta, p)$ never buys. Theorem 4 summarizes the behavioral rule of informed consumers and Figure 7 illustrates their equilibrium strategy in the case that $n = 4$.

Theorem 4 *(i) For* $i \geq 3$*,*

$$\left.\begin{array}{l} \textit{If } v_{i-1} \geq p \\ \textit{If } v_{i-1} \in [p - r + \delta, p) \\ \textit{If } v_{i-1} < p - r + \delta \end{array}\right\} \textit{Consumer } i \textit{ will} \left\{\begin{array}{l} \textit{buy and refer if } v_i > p - r + \delta \\ \textit{buy and refer if } v_i \geq p - \hat{\alpha}_{i-1} r + \delta \\ \textit{buy buy and not refer if } v_i \geq p. \end{array}\right.$$

(ii)

$$\left.\begin{array}{l} \textit{If } v_1 \geq p \\ \textit{If } v_1 < p \end{array}\right\} \textit{Consumer 2 will} \left\{\begin{array}{ll} \textit{buy and refer} & \textit{iff } v_2 \geq p - r + \delta \\ \textit{buy and not refer} & \textit{iff } v_2 \geq p. \end{array}\right.$$

(iii) Consumer 1 will buy if and only if $v_1 \geq p$*.*

[11] This is justified if δ is negligible.

6.2 Strategies of the Monopolist

Although each consumer makes a buying and referral decision based on his information, the monopolist cannot observe the local information each consumer possesses. Since $\hat{\alpha}_{i-1}(v_{i-1})$ and $\hat{v}_{i+1}(\hat{\alpha}_{i-1})$ depend on the value of v_{i-1} for all $i \geq 4$, the monopolist must take account of all possibilities of realizations of $v_{i-1}(i \geq 4)$. This makes the formula of the monopolist's profit too complicated, although it is not impossible to find the explicit formula. For an illustrative purpose, the profit when $n = 2$ is given below as

$$\pi(2, \delta) = (p - c)(1 - F(p))F(p) + (2(p - c) - r)(1 - F(p - r + \delta))(1 - F(p)),$$

and the monopolist's profit for $n = 3$ is presented in Appendix.[12]

Table 1 compares the simulation outcomes of network transactions in the case of informed and uninformed consumers, for $c = .01$ and $\delta = .005$ under the assumption of uniform distribution. It shows that both profits and social welfares are higher when consumers observe their neighbor's valuations. The intuition is as follows. When consumers are uninformed of their neighbors' valuations, more referral costs are wasted, and thus the monopolist must compensate consumers for bearing high expected referral costs, thereby reducing profits. When consumers are informed of neighbors' valuations, such losses from unsuccessful referrals will be reduced, albeit not eliminated, since they do not refer at all if they know that their neighbor's valuation is so low as to be unlikely to buy.

Finally, Figure 8 and 9 contrast the ranges of over- and under-utilization of social networks in the case of uninformed and informed consumers for $c = .01$.

6.3 Discussion on the Buy-to-Refer Constraint

If a consumer can refer his neighbor without buying one himself, his buying decision depends only on his valuation, v_i, and his referral decision depends only on v_{i-1} (not on v_{i-2}), because the buying decision of consumer $i - 1$ does not depend on v_{i-2} at all. Specifically, consumer i buys if and only if $v_i \geq p$ and refers only if $v_{i-1} > p$.

Note that, if a consumer (with the exception of the first consumer) does not buy, the next consumer never buys. The reason is as follows. The fact that consumer $i(\neq n)$ does not

[12] A general expression for the profit can be obtained from authors upon request.

buy implies that $v_i < p$, in turn implying that he is not referred. Thus, there is no way that consumer $i-1$ will get informed of the product so that he cannot buy one. This suggests that the dependency of consumers' buying behavior is preserved in this case.

Let us compute the monopolist's profit. There are two cases in which exactly m consumers out of n consumers buy. In one case, first m consumers buy. In the other case, consumer n refers without buying and the next m consumers buy. Hence, the monopolist's profit is

$$\pi(n) = \sum_{m=1}^{n} \pi_m,$$

where

$$\pi_m = [m(p-c) - (m-1)r][1-F(p)]^m F(p) + m(p-c-r)[1-F(p)]^m F(p)^a,$$

where $a = 2$ for $1 \leq m < n-2$ and $a = 1$ for $m = n-1$ and

$$\pi_n = [n(p-c) - (n-1)r][1-F(p)]^n.$$

Figure 10 compares the profits with and without the buy-to-refer constraint. The area in which the profit with the constraint exceeds the profit without it is represented by Region B.

7 Correlated Valuations

The sections above assume that v_i is i.i.d. In many situations, however, consumers' valuations are correlated. In other words, a high v_i indicates that v_j is likely to be high for $j \neq i$. To model this situation, we assume that each consumer has a valuation which is a random deviation from the population-average valuation. In other words,

$$v_i = \theta + \varepsilon_i, \tag{5}$$

where θ is the unobservable population-average valuation, and ε_i is the deviation of consumer i's valuation from the population average. We assume that ε_i is i.i.d. with mean zero and the same variance across all i. As before, v_i is learned before consumer i buys.[13]

[13] If consumers can learn v_i only after they buy, the problem becomes trivial, and the monopolist will charge the price of θ to all consumers.

In this situation, it is crucial for consumer i to make a correct inference of v_{i-1} for his referral decision. Since v_i's are correlated, he can infer it on the basis of his own information v_i and the fact that he is referred i.e. all his predecessors have referred their successor.

We will denote by Ω_i the information contained in the fact that consumer i is referred. Also, let $\tilde{v}_i$ and $\hat{v}_i$ be the minimum valuation required for consumer i to buy and to refer, respectively.[14] Then, $\hat{v}_i$ must satisfy the following:

$$r\left(1 - F\left(\tilde{v}_{i-1}|\hat{v}_{i,}\Omega_i\right)\right) = \delta, \tag{6}$$

where

$$\Omega_i = \left\{v_k > \hat{v}_k \text{ for all } k > i\right\}, \text{ and } \Omega_n = \emptyset. \tag{7}$$

Also, the definition of $\tilde{v}_i$ implies that

$$\tilde{v}_i = p - \max\left\{r\left(1 - F\left(\tilde{v}_{i-1}|\tilde{v}_{i,}\Omega_i\right)\right) - \delta, 0\right\}. \tag{8}$$

Equations (6) and (8) lead to the following proposition.

Proposition 9 *$\tilde{v}_i = p$ for all $i \in N$.*

Proof. See the appendix.

The intuition is as follows. By definition, a consumer whose valuation is $\hat{v}_i$ expects no net gain from making a referral. Thus, all consumers whose valuation is $v_i \leq \hat{v}_i$ will not refer even if they may buy. This implies that a consumer buys only when $v_i \geq p$.

One implication of Proposition 9 is that a consumer who is indifferent between buying and not buying does not refer. This is because his benefit from the referral service is less than the referral cost. However, this does not imply that a consumer does not refer at all for any v_i, unlike in the case of independent valuations. The difference comes from a positive correlation between v_i and v_{i-1}. Since the probability that the referred successor will buy is increasing in v_i in this correlated case, a higher v_i than p is required for consumer i to refer consumer $i-1$.

Also, equation (6) and Proposition 9 lead to the following for $\hat{v}_i$:

$$r\left(1 - F\left(p|\hat{v}_{i,}\Omega_i\right)\right) = \delta \text{ for all } i \in N. \tag{9}$$

[14] There exists $\hat{v}_i$ if δ is high enough.

Inspecting the structure of Ω_i, we can see that $\hat{v}_i$ can be determined recursively from $\hat{v}_n$, not as the minimum valuation was determined from $\tilde{v}_1$ in the case of independent valuations. From equation (9), we have the following proposition.

Proposition 10 *$\hat{v}_i$ increases in i for all $i \in N \setminus \{1\}$.*

Proof. See the appendix.

This proposition implies that a consumer is more likely to refer, as the chain of referrals continues, contrary to Section 4. The intuitive reason for this is that, if a consumer located later in the chain, is referred, it implies that more predecessors have high valuations, leading him to conclude that the next consumer is more likely to buy.

The argument hitherto demonstrates that a consumer's referral decision after a purchase depends on his own valuation,[15] contrary to Section 4.[16] Only consumers with high valuation refer his neighbor because he believes that the neighbor's valuation is also likely to be high. This outcome seems to fit the referral pattern in the real world better.[17]

8 Conclusions

In this paper, we have presented a model of transactions over the social network and characterized the optimal price and referral fee. We conclude that, compared to the social optimum, both over- and under-utilization of the social network are possible.

The role of social networks in economic life has been studied by (economic) sociologists for decades,[18] while economists have begun to pay attention to it only recently.[19] Their

[15]In fact, this is not the only possible outcome. There may be other equilibria in which all consumers who buy refer.

[16]In Section 6, we argued the possibility that a consumer's referral decision might depend on the neighbor's valuation.

[17]For empirical evidences, see Zeithaml *et al.* (1996).

[18]See Barnes (1972) and Coleman (1988) for the sociologists' view on the social network.

[19]Inarguably, labor economists have applied the idea of the network most successfully. For example, the role of referrals based on the old boy network in the job search, earnings and the tenure decision is extensively studied. See, for example, Holzer (1987), Saloner (1985), Montgomery (1991), and Simon and Warner (1992).

studies are mainly concerned about the implications of the given social network,[20] rather than the formation of networks.[21] In this sense, this paper shares with the long tradition of economic sociology and manifests the insight of Granovetter (1985) that *economic* actions are embedded in *social* networks.

There are numerous directions to which this research can be extended. First, we can extend our analysis to a much more complex graph.[22] Suppose that the graph is regular of degree $d \geq n/2$. Since such a regular graph has a path from node n passing through all other nodes only once,[23] all consumers have exactly one predecessor and one successor on the path (except the first consumer and the potential last consumer on the path). Therefore, all the results of the previous sections will be carried over to any regular graph of degree $d \geq n/2$.[24] What if multiple referrals are allowed? The outcome in network-embedded transactions must be closer to the market outcome as the degree is higher, since more links can speed up the diffusion of the product. However, we can show that even a complete graph, in which all consumers (not the monopolist) communicate with one another, cannot duplicate the outcome of the market transaction because the dependency can be completely removed only at the expense of a sufficiently high referral fee.

Second, it would be interesting to give the monopolist more flexibility in its choice. For instance, we may allow the monopolist to choose the optimal number of direct links with the group of consumers. Then, it will trade off the costs of linking with an additional

[20]See Power and Smith-Doerr (1994) and Nohria and Eccels (1992) for a review of the literature on economic sociology.

[21]Theoretical researches on the network formation primarily examine the efficiency and stability of networks. See, for example, Bala and Goyal (1999, 2000), Dutta and Mutuswami (1997), Dutta et al. (1998), Goyal and Vega-Redondo (2000), and Jackson and Wolinsky (1996). For applied works of the network formation, see Hendricks et al. (1995, 1999), Goyal and Joshi (2001), Goyal and Moraga-Gonzalez (2001), and Kranton and Minehart (2000).

[22]See Jun and Kim (2003) for consumer referral in a small world network.

[23]Such a path is called Hamiltonian. A few sufficient conditions for the existence of a Hamiltonian path have been provided. This statement is due to the well known Dirac's Theorem among them. Originally, it says that a graph g with $n \geq 3$ is Hamiltonian if $d_i \geq n/2$ for all $i \in N$ (see West 1996). If a graph is not Hamiltonian but just connected, a consumer may be referred more than once, which may cause some problem.

[24]We note that the regular graph of $d \geq n/2$ is a very special case in the real world, since in many cases, individuals in a network have the much smaller number of neighbors, compared to the size of the population.

consumer against the benefit from it (reducing the dependency of consumers' behavior). We may also allow the monopolist to offer differential referral fees. Clearly, this type of price discrimination will increase the monopolist's profit. We can expect that the monopolist will pay a higher referral fee to a consumer who is earlier in the chain of the network, because the earlier consumer is more valuable to the monopolist, and thus having him to refer is more critical to the monopolist.

Third, it would be worthwhile to incorporate consumer satisfaction into referral decisions. Consumers in our model do not receive any new information once they purchase. However, it is reasonable to expect that after actual consumption occurs, some consumers may feel satisfied or even delighted, while others may be disappointed about the product. A delighted consumer can be modeled as one who obtained a positive surplus from the consumption more than a certain threshold level. Evidence suggests that delighted consumers will be more willing to recommend the service they enjoyed (See Oliver *et al.* (1997).). This will enlarge the disparity between buying and referral decisions.

Finally, it will be intriguing to consider competition between firms. Paying a referral fee is in many instances the consequence of strategic behavior in the presence of competition between firms. This issue will be addressed in our forthcoming paper.

9 Appendix

Proof of Proposition 1

We will prove this by induction. Suppose $\tilde{v}_k = p$ and $\tilde{v}_{k+1} < p$ for some k, where

$$\tilde{v}_{k+1} = p - \max\{r[1 - F(\tilde{v}_k)] - \delta, 0\}. \tag{10}$$

This implies that

$$r[1 - F(\tilde{v}_{k+1})] > r[1 - F(\tilde{v}_k)] > \delta. \tag{11}$$

Consider the minimum valuation of consumer $k + 2$:

$$\tilde{v}_{k+2} = p - \max\{r[1 - F(\tilde{v}_{k+1})] - \delta, 0\}. \tag{12}$$

Due to inequality (11), we can establish that $\tilde{v}_{k+2} < \tilde{v}_{k+1}$.

Now suppose $\tilde{v}_{m+1} < \tilde{v}_m$ for any m. We will show that $\tilde{v}_{m+2} < \tilde{v}_{m+1}$. The minimum valuations of consumer $m+1$ and $m+2$ are as follows:

$$\tilde{v}_{m+1} = p - \max\{r[1 - F(\tilde{v}_m)] - \delta, 0\}. \tag{13}$$

$$\tilde{v}_{m+2} = p - \max\{r[1 - F(\tilde{v}_{m+1})] - \delta, 0\}. \tag{14}$$

Since $\tilde{v}_m > \tilde{v}_{m+1} \Longrightarrow r[1 - F(\tilde{v}_{m+1})] > r[1 - F(\tilde{v}_m)] > \delta$, (13) and (14) imply that $\tilde{v}_{m+2} < \tilde{v}_{m+1}$. This completes the induction.

Proof of Proposition 2

It is direct from Proposition 1 that if $\tilde{v}_2 < p$, $\tilde{v}_n < \tilde{v}_{n-1} < \tilde{v}_{n-2} < ... < \tilde{v}_3 < p$. It remains to prove the converse. We will prove by contradiction. Suppose $\tilde{v}_2 = p$. From equation (3), we have

$$\tilde{v}_3 < p \Longrightarrow r\left(1 - F\left(\tilde{v}_2\right)\right) > \delta \Longleftrightarrow p(1 - F(p)) > \delta.$$

This implies that $\tilde{v}_2 < p$ from equation (2), which is a contradiction.

Proof of Theorem 2

We start with the following observation:

$$\pi_N^*\left(2, 0\right) > \pi_0^*, \tag{15}$$

since, for any $r \in (0, p_M^* - c)$,

$$\begin{aligned} \pi_N^*\left(2, 0\right) &\geq \left(p_M^* - c\right)\left(1 - F\left(\tilde{v}_2\right)\right) + \left(p_M^* - c - r\right)\left(1 - F\left(\tilde{v}_2\right)\right)\left(1 - F\left(p_M^*\right)\right) \\ &> \left(p_M^* - c\right)\left(1 - F\left(p_M^*\right)\right) \equiv \pi_0^* \end{aligned}$$

where the first inequality is due to the optimality condition of $\pi_N^*\left(2, 0\right)$. Assuming concavity of $\pi_N\left(2, \delta\right)$, $\pi_N^*\left(2, \delta\right)$ is continuous with respect to δ and clearly $\lim_{\delta \to \infty} \pi_N^*\left(2, \delta\right) = \pi_0^*$. Note that $\pi_N^*\left(n, \delta\right)$ is nondecreasing in δ. This and equation (15) imply that there exists $\underline{\delta}\left(2\right)$ such that $\pi_N^*\left(2, \delta\right) > \pi_0^*$ for all $\delta < \underline{\delta}\left(2\right)$ and $\pi_N^*\left(2, \delta\right) = \pi_0^*$ for all $\delta \geq \underline{\delta}\left(2\right)$. This completes the proof when $n = 2$. To prove the theorem for general n, we first note that

$$\pi_N^*\left(n+1, 0\right) > \pi_N^*\left(n, 0\right) \text{ for all } n \geq 2.$$

This is direct from the definition of the profit function. This implies $\pi_N^*\left(n, 0\right) > \pi_0^*$ for all n. By Theorem 1 and Proposition 2, the result is immediate for all $n \geq 2$.

Proof of Theorem 3

Since it is clear that $\pi_N^* < \pi_M^*$ for $\delta > \underline{\delta}(n)$, it is sufficient to show that $\pi_N^* < \pi_M^*$ for $\delta \leq \underline{\delta}(n)$. Since [RC] is satisfied for $\delta \leq \underline{\delta}(n)$, we get:

$$\begin{aligned}
\pi_N^* &= (p_N^* - c)(1 - F(p_N^*)) \prod_{i=2}^{n} (1 - F(\tilde{v}_i)) + \sum_{k=2}^{n} [p_N^* - c - r_N^* (1 - F(\tilde{v}_{k-1}))] \prod_{i=k}^{n} (1 - F(\tilde{v}_i)) \\
&< \max_p (p - c)(1 - F(p)) + \max_{\tilde{v}_k} \sum_{k=2}^{n} [\tilde{v}_k - \delta - c] \prod_{i=k}^{n} (1 - F(\tilde{v}_i)) \\
&< (p_M^* - c)(1 - F(p_M^*)) + (n-1)(p_M^* - c)(1 - F(p_M^*)) \\
&= \pi_M^*.
\end{aligned}$$

Proof of Proposition 3

If consumer i whose valuation is v_i buys in equilibrium, $v_i - p + \max\{\hat{\alpha}_{i-1} r - \delta, 0\} \geq 0$. Since $v_i' > v_i$, it follows that $v_i' - p + \max\{\hat{\alpha}_{i-1} r - \delta, 0\} \geq 0$. This implies that consumer i whose valuation is v_i' buys in equilibrium.

Proof of Proposition 5

The first part is clear. If consumer i buys, then he will refer from the first part. This implies that

$$\max\{\hat{\alpha}_{i-1} r - \delta, 0\} = \hat{\alpha}_{i-1} r - \delta.$$

Therefore his payoff is $v_i - p + \hat{\alpha}_{i-1} r - \delta$. Hence he will buy if and only if $v_i \geq p - \hat{\alpha}_{i-1} r + \delta$.

Proof of Proposition 7

It is easy to see from Proposition 6 that $\hat{\alpha}_2(v) \geq \hat{\alpha}_1(v)$ for all v. Suppose that $\hat{\alpha}_i(v) \geq \hat{\alpha}_{i-1}(v)$.

$$\begin{aligned}
\hat{\alpha}_{i+1}(v) &= \int_{\underline{v}}^{p-r+\delta} I_{\{v \geq p\}} dF(v) + \int_{p-r+\delta}^{p} I_{\{v \geq p - \hat{\alpha}_i(v) r + \delta\}} dF(v) + \int_{p}^{\bar{v}} I_{\{v \geq p - r + \delta\}} dF(v) \\
&= \int_{p-r+\delta}^{p} I_{\{v \geq p - \hat{\alpha}_i(v) r + \delta\}} dF(v) + 1 - F(p) \\
&\geq \int_{p-r+\delta}^{p} I_{\{v \geq p - \hat{\alpha}_{i-1}(v) r + \delta\}} dF(v) + 1 - F(p) \\
&= \hat{\alpha}_i(v).
\end{aligned}$$

This implies that $\hat{\alpha}_{i+1}(v) \geq \hat{\alpha}_i(v)$ for all i. The second result directly follows from this.

Proof of Proposition 8

[RC] implies that $\max\{\hat{\alpha}_2 r - \delta, 0\} = \hat{\alpha}_2 r - \delta$. Therefore the results follows from Proposition 7.

Monopolist Profit for $n = 3$ in the Case of Informed Consumers

Suppose there are three consumers, $n = 3$. There are three separate cases to consider. If $v_2 > p$, consumer 3 will buy and refer if $v_3 > p - r + \delta$. If consumer 1 buy, all consumers purchase. If consumer 1 does not purchase, only consumer 2 and 3 will purchase. Hence the profit in this case is

$$\begin{aligned}\pi_1 &= (1 - F(p - r + \delta))(1 - F(p))^2 (3(p - c) - 2r) \\ &+ (1 - F(p - r + \delta))(1 - F(p)) F(p)(2(p - c) - r).\end{aligned}$$

If $v_2 < p - r + \delta$, consumer 3 will buy (and not refer) if $v_3 > p$. Therefore only consumer 3 will buy if $v_3 > p$. Hence the profit in this case is

$$\pi_2 = (1 - F(p)) F(p - r + \delta)(p - c).$$

If $p - r + \delta \leq v_2 < p$, consumer 3 knows that consumer 2 will buy if and only if consumer 1 buys. Hence consumer 3 will buy and refer if and only if $v_3 \geq p - r(1 - p) + \delta$.[25]

$$\begin{aligned}\pi_3 &= (1 - F(p - r(1 - p) + \delta))(F(p) - F(p - r + \delta))(1 - F(p))(3(p - c) - 2r) \\ &+ (1 - F(p - r(1 - p) + \delta))(F(p) - F(p - r + \delta)) F(p)(p - c).\end{aligned}$$

Therefore, for $n = 3$, the overall monopolist profit is $\pi(3, \delta) = \pi_1 + \pi_2 + \pi_3$. $\|$

Proof of Proposition 9

Considering that $\tilde{v}_i \leq \hat{v}_i$ it follows from (6) and (8) that

$$\max\{r(1 - F(\tilde{v}_{i-1}|\tilde{v}_i, \Omega_i)) - \delta, 0\} = 0,$$

since $r(1 - F(\tilde{v}_{i-1}|\tilde{v}_i, \Omega_i)) \leq r(1 - F(\tilde{v}_{i-1}|\hat{v}_i, \Omega_i))$. This implies that $\tilde{v}_i = p$.

[25] If $v_3 \geq p$, buying only gives consumer 3 the net benefit of $v_3 - p$, which is smaller than the net benefit of buying and referring, $v_3 - p + r(1 - p) - \delta$. If $p + r(1 - p) - \delta \leq v_3 < p$, consumer 3 will also buy and refer, since the net benefit is positive.

Proof of Proposition 10

Given the definition of Ω_i in (7), $1-F\left(\tilde{v}_{i-1}|v_{i,}\Omega_i\right)$ is decreasing in i for all v and increasing in v. The result is immediate.

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Figures and Table in the Text

Table 1. Comparison of Unformed and Informed Consumers

	$n = 2$		$n = 3$	
	Uninformed	Informed	Uninformed	Informed
Price	.505	.532	.515	.524
Referral Fee	.258	.291	.311	.289
Profits	.378	.396	.464	.491
Social Welfares	.641	.666	.875	.913

Figure 1. Locus of Price and Referral Fee as n changes: c = 0.27, delta = 0.05

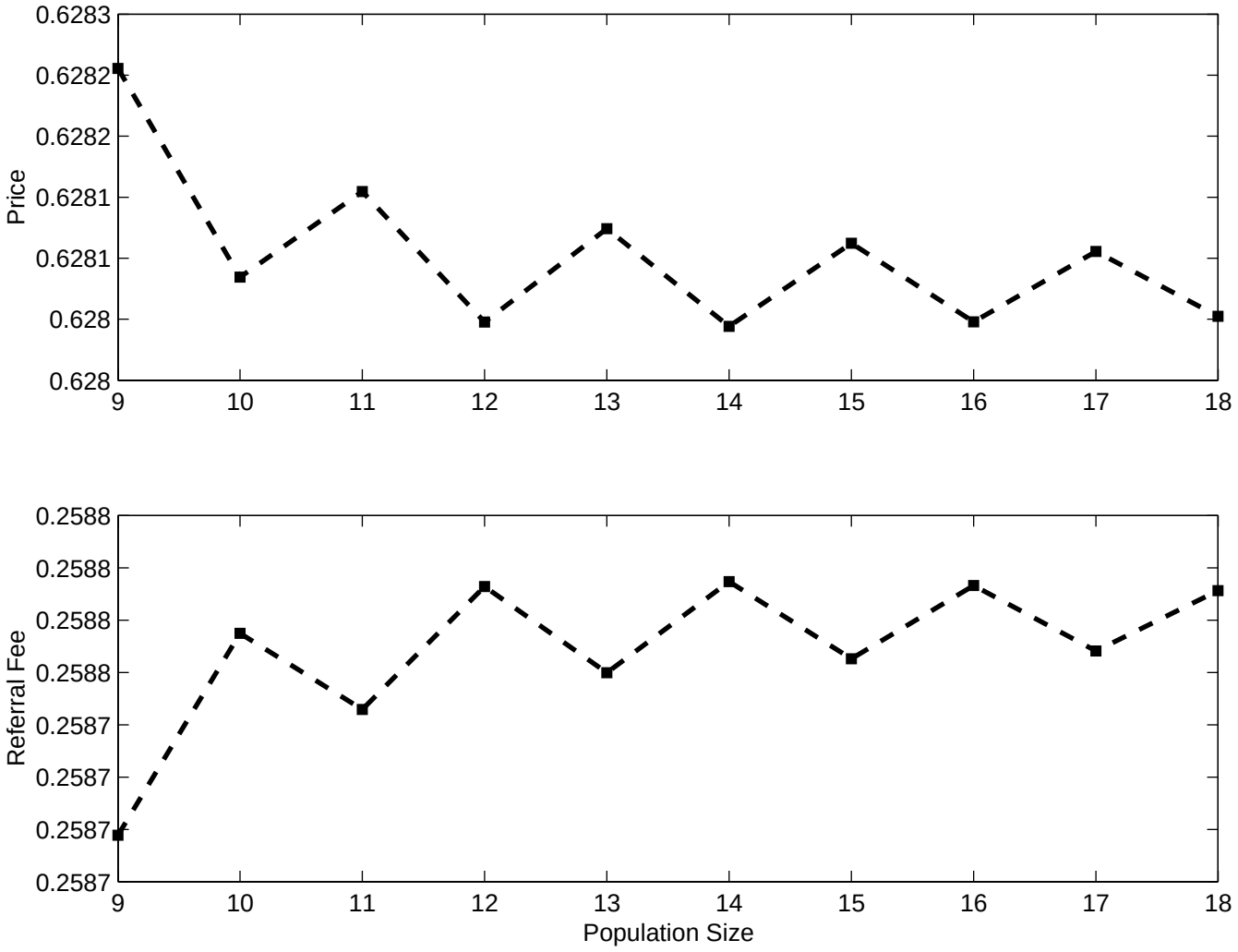

Figure 2. Locus of Price and Referral Fee as c changes: n = 12, delta = 0.001

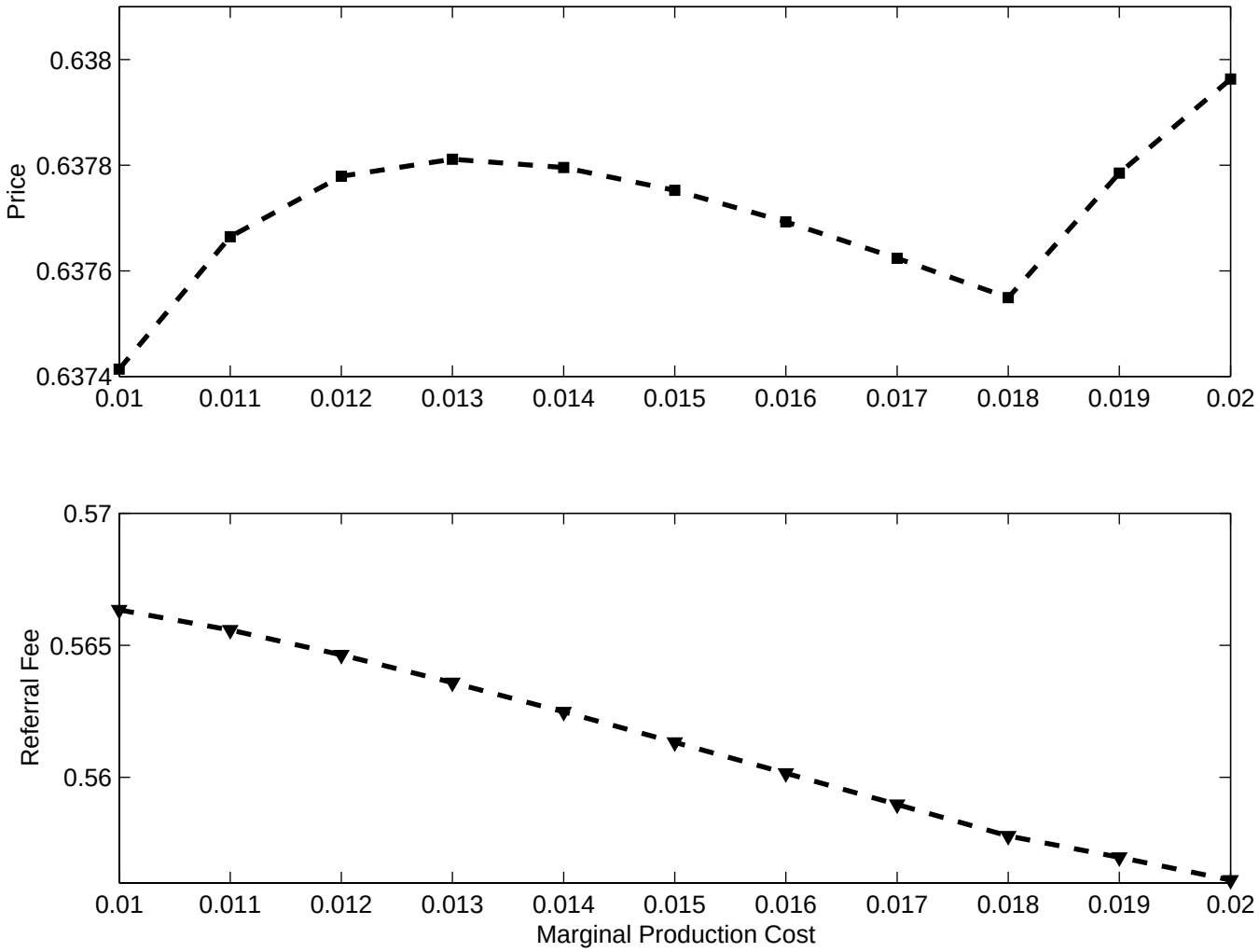

Figure 3. Locus of Price and Referral Fee as delta changes: n = 10, c = 0.2

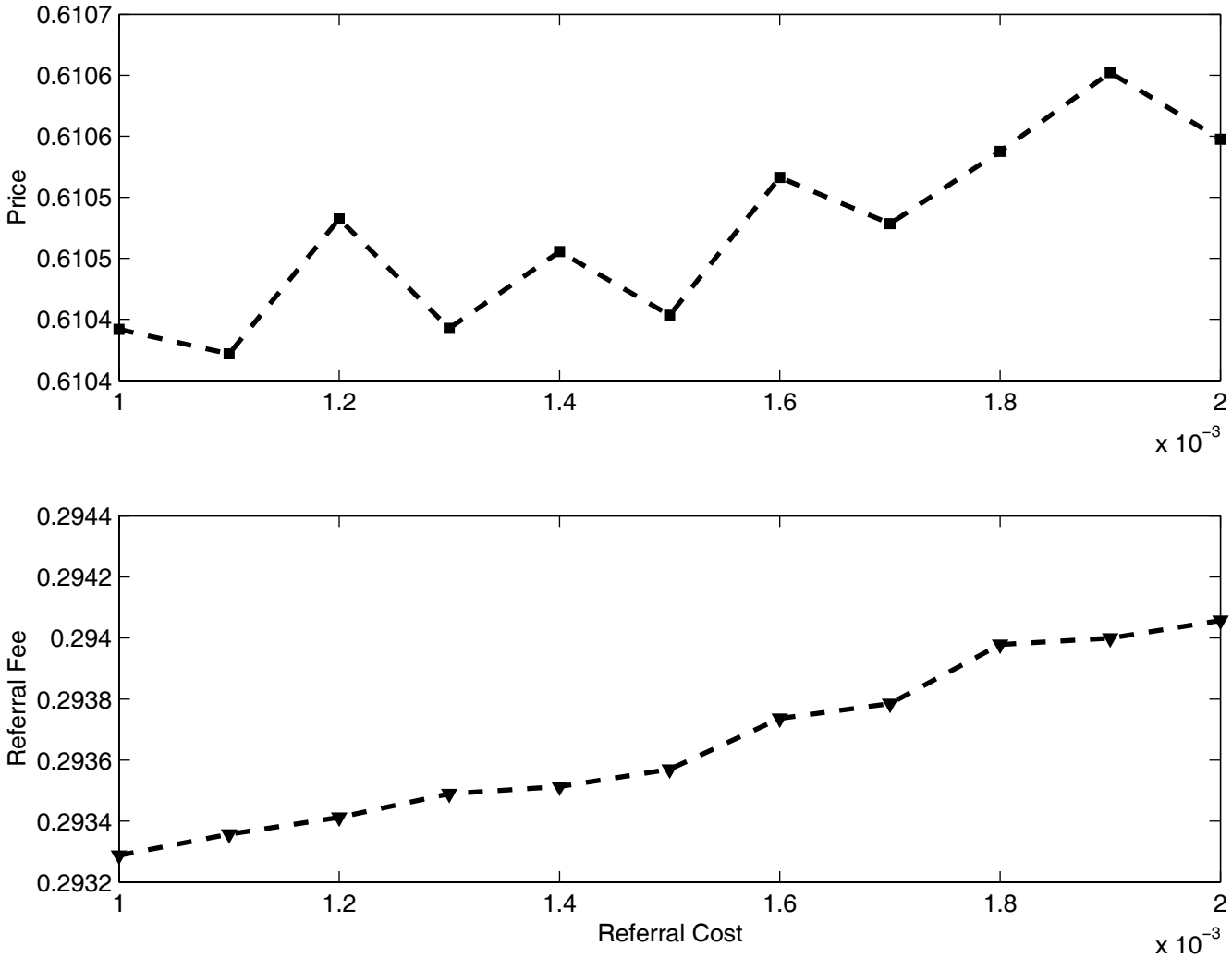

Figure 4. Profits with and without the Buy−to−Refer Constraint: Case of Uninformed Consumers

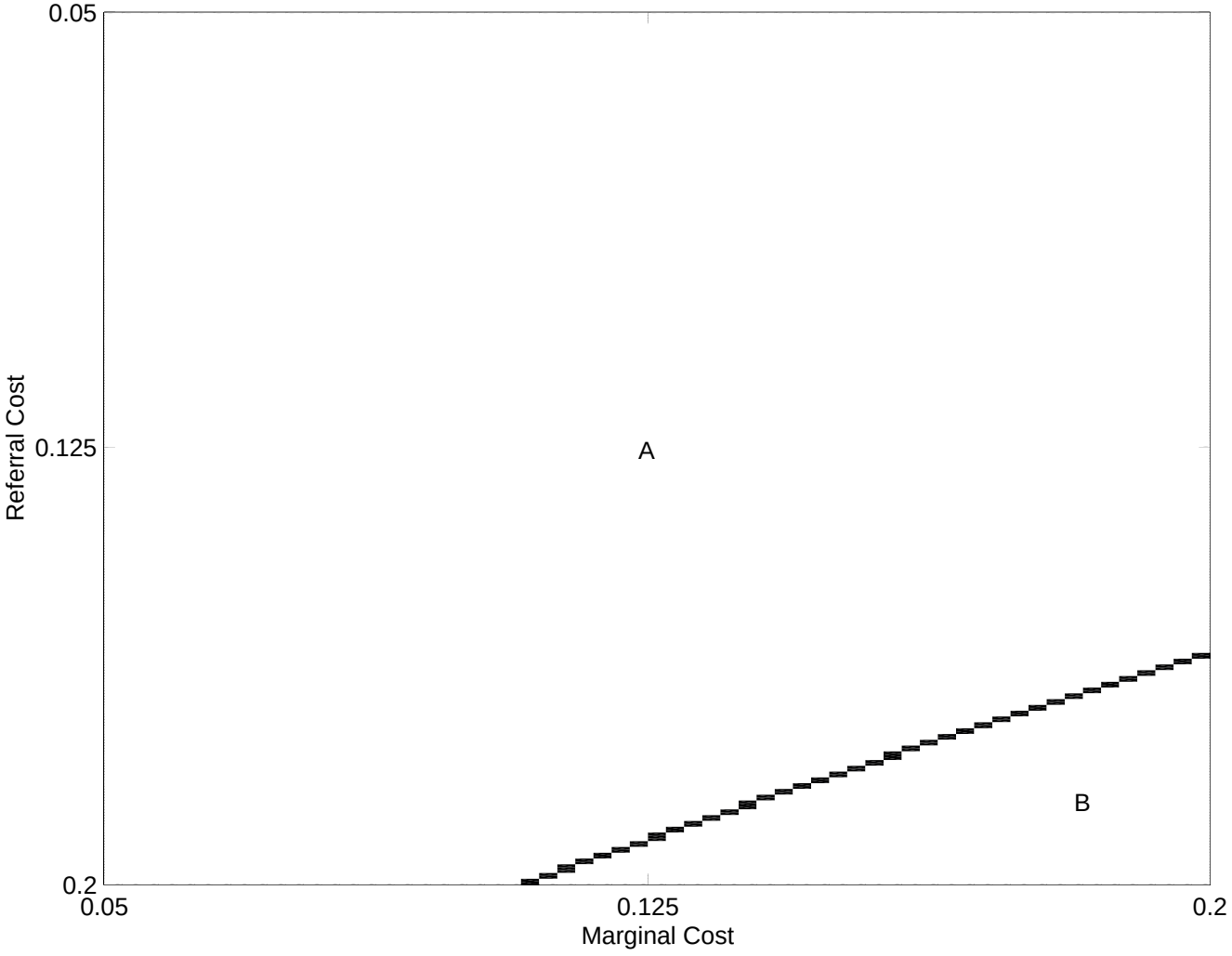

Figure 5. Uninformed Consumer Network vs. Market: Profits and Welfares for n = 2

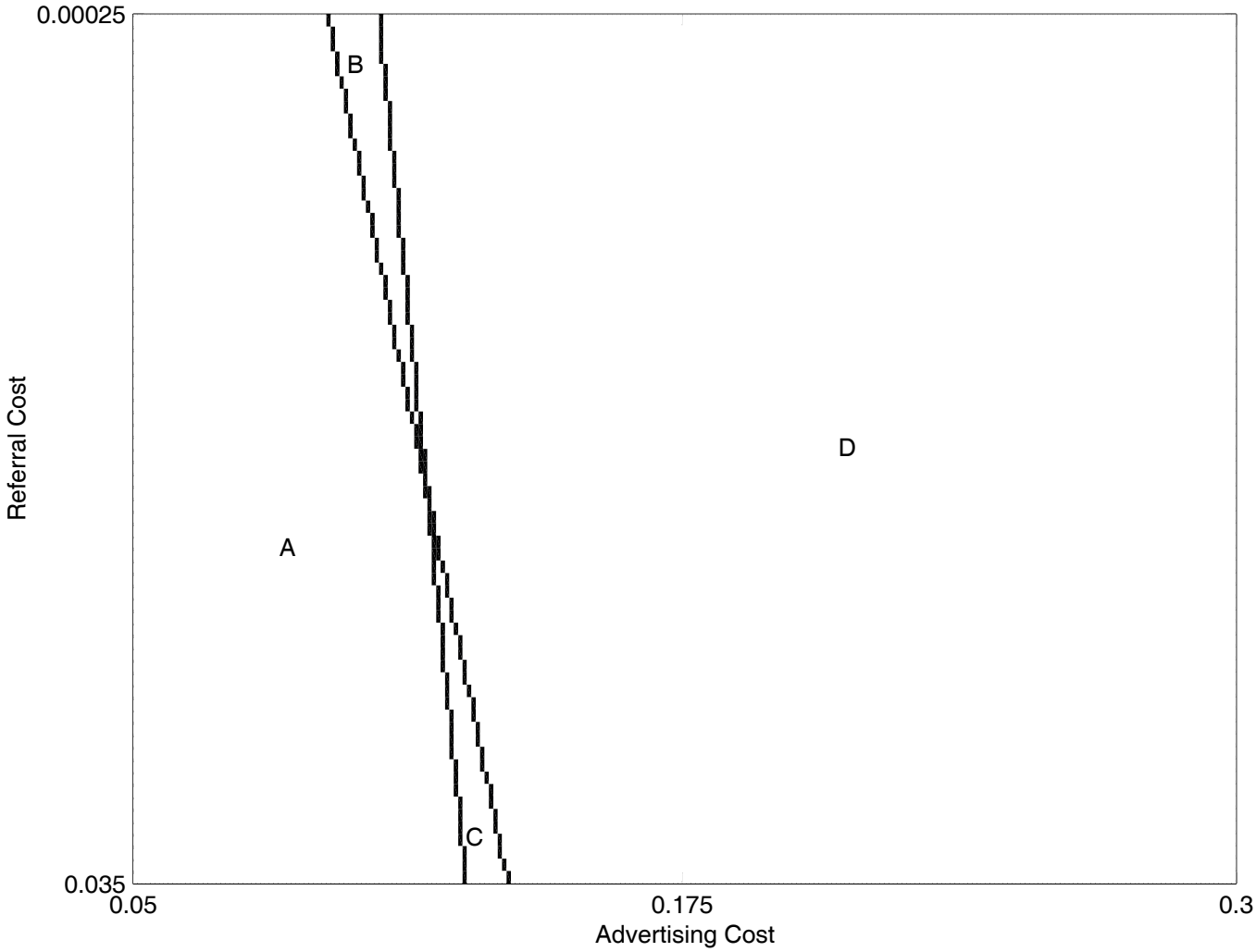

Figure 6. Uninformed Consumer Network vs. Market: Profits and Welfares for n = 3

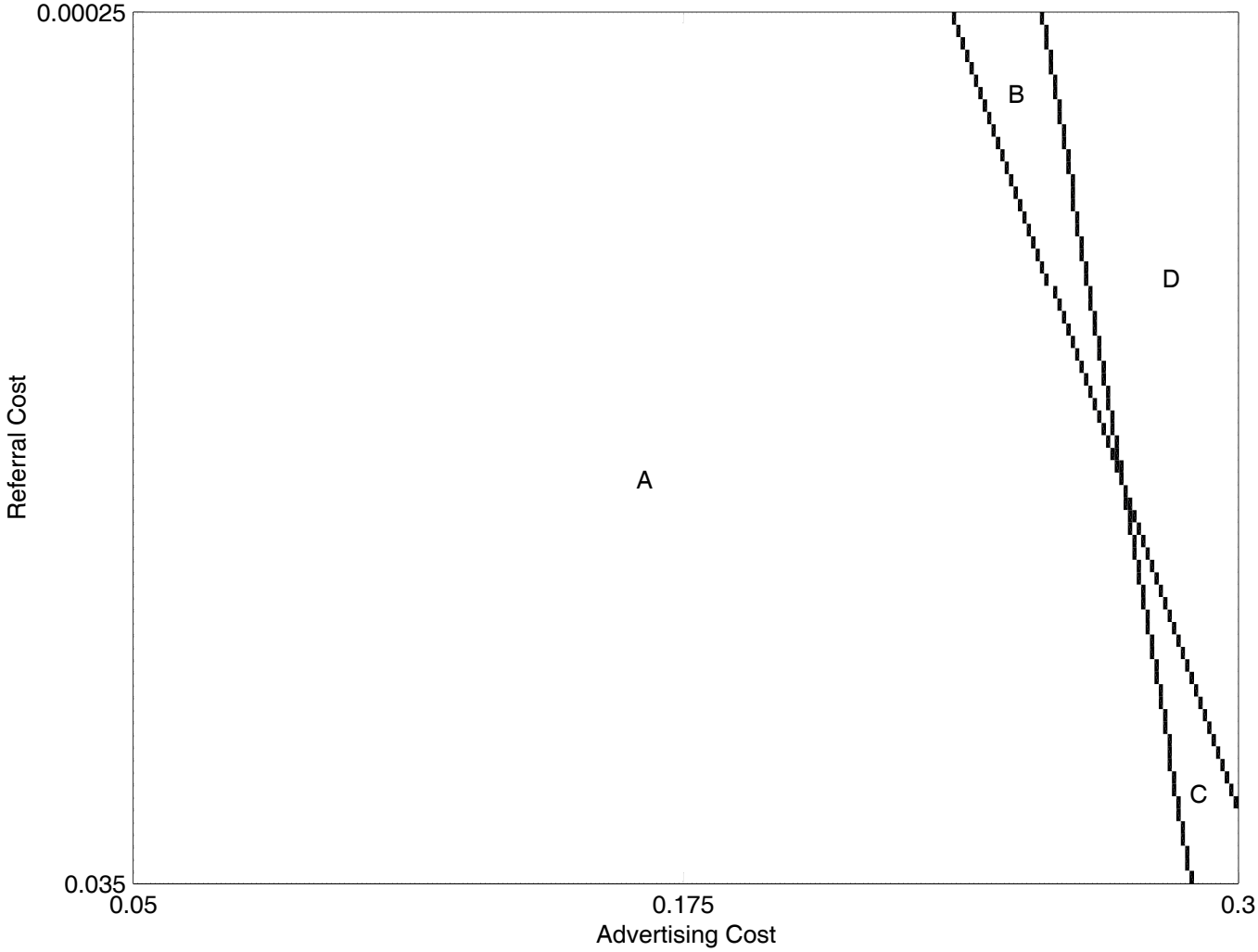

Figure 7: Strategy of Informed Consumers when $n = 4$

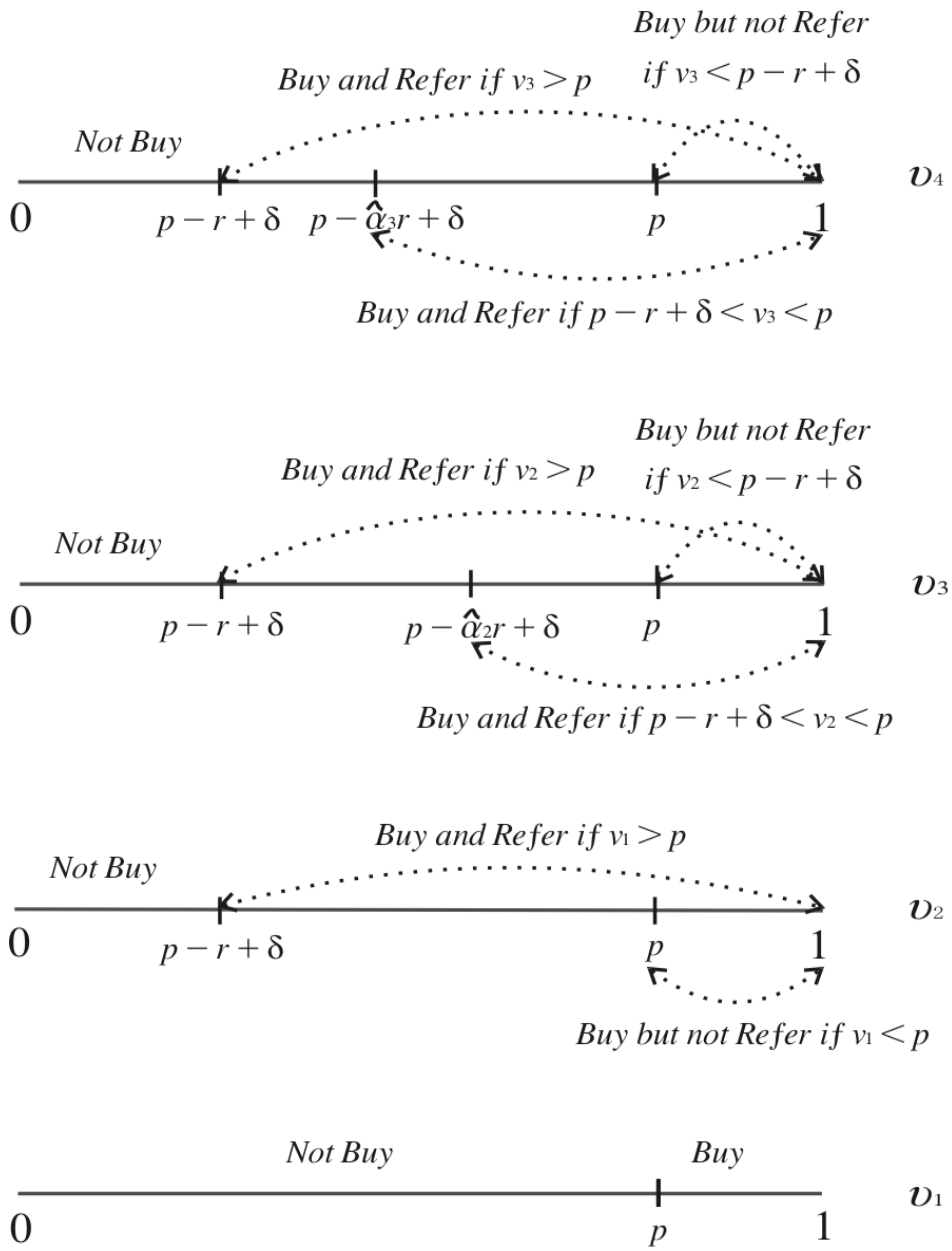

Figure 8. Informed Consumer Network vs. Market: Profits and Welfares for n = 2

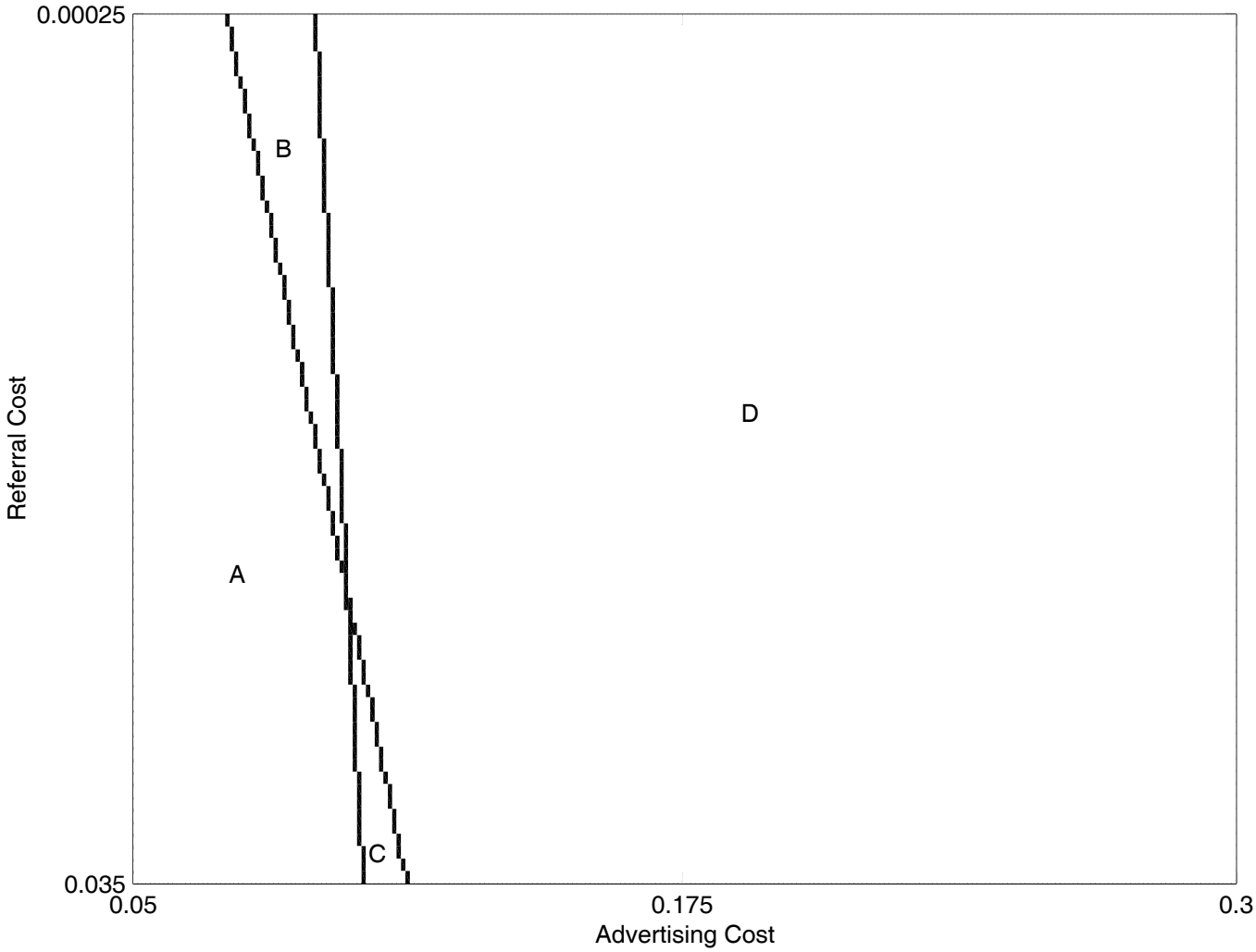

Figure 9. Informed Consumer Network vs. Market: Profits and Welfares for n = 3

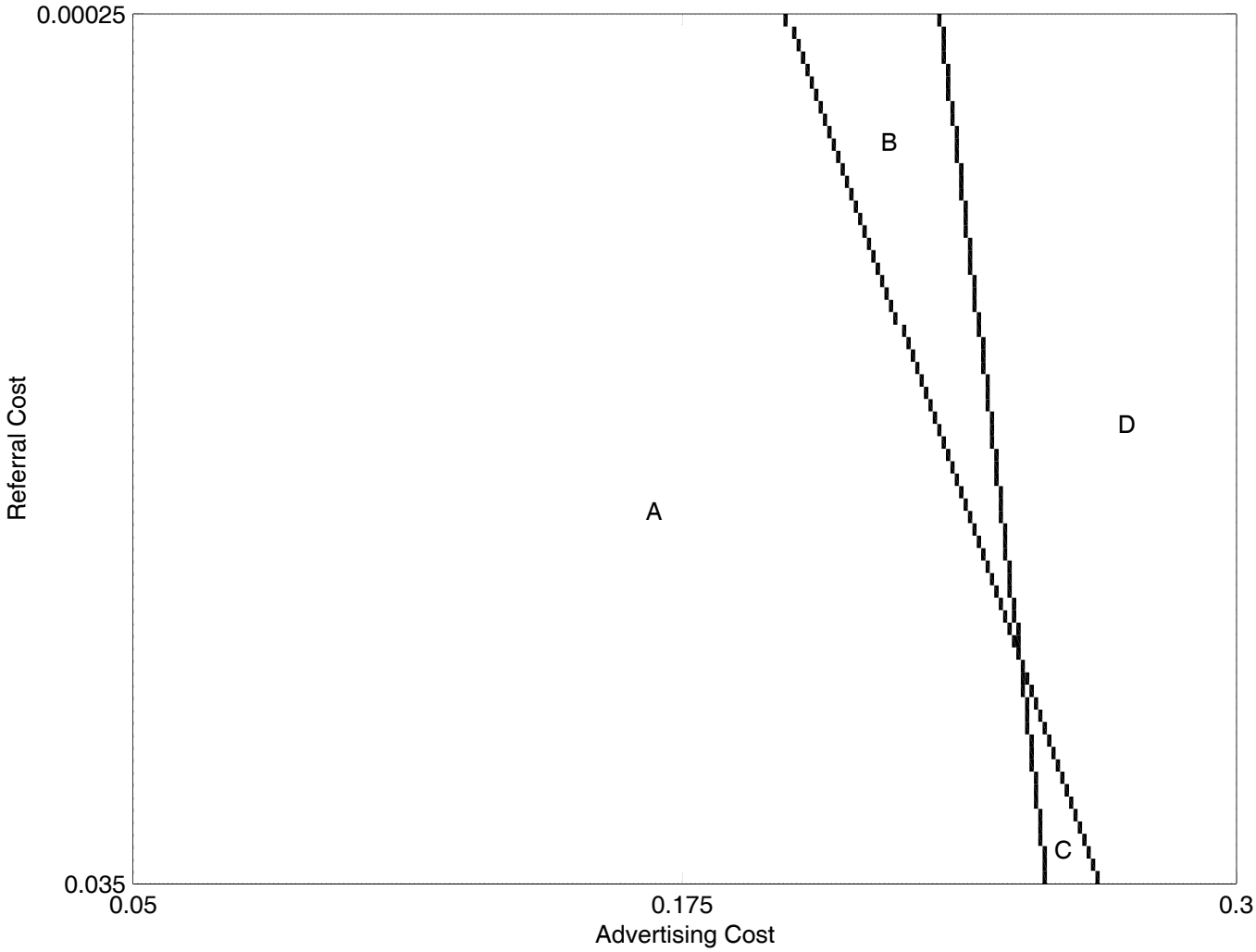

Figure 10. Profits with and without the Buy−to−Refer Constraint: Case of Informed Consumers