Connectivity and Hubs in a Network as an Information Flow

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Abstract

We consider the problem of forming a network as an information flow under the requirement that the whole network be connected and remain connected after the destruction of some nodes. We introduce the $\mathcal{C}^q$-stability and the $\mathcal{C}^q$-efficiency of a network which, roughly speaking, require connectivity in addition to stability and efficiency even after any q nodes are destroyed. We mainly examine the relation between efficiency and stability. With the connectivity requirement, the efficient network is always stable, but the $\mathcal{C}^q$-efficient network is not necessarily $\mathcal{C}^q$-stable for $q \geq 1$. We provide sufficient conditions for a $\mathcal{C}^q$-efficient network to be $\mathcal{C}^q$-stable.

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1 Introduction

Many economic interactions rely on social networks. Just as physical networks like telecommunications network or Internet do, social networks based on solid interpersonal relationships enable or facilitate exchange of valuable information or transactions among agents on a network. Thus, understanding the structure of networks has been recognized as important to explaining many natural or economic/social phenomena.

There has been a significant development in the theory of network formation since the seminal article by Jackson and Wolinsky (1996).[1] This article departs from the literature in the following senses: (1) The role of a network is to communicate information and each agent minimizes the sum of the cost of connecting direct links and the delay cost caused by indirect links.[2] (2) The whole network must be connected. (3) An agent (or agents) may die and the network must remain connected after the death of some agents.

Connectivity is important in networks such as Internet, airline network or electricity network. Especially, in operating the Internet business, it is essential to maintain universal connectivity which can be achieved by independent Internet service providers (nodes). Besides, it is quite often that some nodes in a network are destroyed by various external shocks, and thus naturally the planner who is in a position of designing the structure of the network is prepared against the possibility. In the social context, too, a person is trying to keep multiple connections with politically influential figures for fear that one of them may lose his power. Despite the common recognition that a network should be robust to outer shocks, no literature on network formation has taken into account the possibility of node destructions.[3] For example, a single star may turn out to be stable and efficient in the term of Jackson and Wolinsky, but its emergence in reality is questionable, because it loses its attractiveness once the center happens to break down.[4]

In this article, we are mainly interested in which networks will emerge when connectivity

[1]To name a few, see Dutta and Mutuswami (1997), Bala and Goyal (2000a), Jackson and van den Nouweland (2005), and Jackson and Watts (2002). For an extensive survey, see Jackson (2003).

[2]This means that the benefit from connections in our model is fixed in a contrast with the connection model of Jackson and Wolinsky.

[3]Bala and Goyal (2000b), and Haller and Sarangi (2003) consider the possibility that a link, rather than a node, may not function.

[4]In physics, there are articles on incorporating the aging phenomenon (that a node loses links by aging) into the scale-free network. See, for example, Amaral *et al.* (2000).

of a network must be maintained even after outer shocks causing the destruction of some nodes. Following the spirit of Jackson and Wolinsky, we introduce the $\mathcal{C}^q$-stability and the $\mathcal{C}^q$-efficiency of a network which, roughly speaking, require connectivity in addition to stability and efficiency even after any q nodes are eliminated.

Although it is usually required for connectivity that a network have more direct links as more nodes are expected to be destroyed, $\mathcal{C}^q$-stability (or $\mathcal{C}^q$-efficiency) does not necessarily imply $\mathcal{C}^{q-1}$-stability ($\mathcal{C}^{q-1}$-efficiency), because $\mathcal{C}^q$-stability (or $\mathcal{C}^q$-efficiency) of a network requires the network to survive fewer deviations than $\mathcal{C}^{q-1}$-stability. Hence, $\mathcal{C}^q$-stability (or $\mathcal{C}^q$-efficiency) is not monotonic with respect to q. Also, in the face of possible shocks, a network like a star which relies on a single center for its connectivity cannot be sustained in terms of either stability or efficiency, because the destruction of the center would make the network totally disconnected. Thus, the network with additional links between noncentered agents or more than one hub is expected to emerge. We demonstrate, however, that a network with more than one perfect hub – which has links to all other nodes – is neither $\mathcal{C}^q$-stable nor $\mathcal{C}^q$-efficient.

We also examine the relation between efficiency and stability. The efficient network is always stable, which is mainly because the connectivity requirement does not allow the possibility of overconnectedness. However, for $q \geq 1$, the $\mathcal{C}^q$-efficient network is not necessarily $\mathcal{C}^q$-stable. We provide sufficient conditions for a $\mathcal{C}^q$-efficient network to be $\mathcal{C}^q$-stable.

The essential role of a hub is to shorten the length of the path between any two nodes in a network. However, on the other hand, it is fragile to external shocks, for example, attacks from the outside.[5] That is, once it is attacked, the whole network structure breaks down into disconnection. This fatality could be circumvented by having multiple hubs. In short, multiple hubs, which are not perfect, emerge against the shocks that can destroy some nodes.

Several formal and informal explanations have been provided for the emergence of "hub" networks, especially in the airline industry. Hendricks *et al.* (1995) attributes it to economies of density in the number of travelers between two directly connected nodes. Also, as Morrison and Winston (1990) report, the high traffic density of the hub-spoke structure enables the airline to supply more frequent flights, thereby attracting more business customers who need flexibility in scheduling. Barabási and Réka (1999) model the scale-free network[6] based on

[5]See Réka *et al.* (2000) for the original discussion on the fragile nature of hub structures to attacks and Newman *et al.* (2000) for the analytic discussion on it.

[6]A scale-free network is a network in which the distribution of connectivity is extremely uneven in the

the evolution and preferential attachment to explain the emergence of a hub and the power law.[7] According to them, a network can have hubs if nodes enter the population sequentially so that earlier comers have higher chances to be linked with others. If new nodes attach a link to a node connected to more links with higher probability, a hub will emerge more rapidly and the network will follow the power law. However, in this paper, the emergence of a hub is not related to the sequential nature of connections nor preferential attachment.[8] Rather, in a separate work, we assert that it is hard to support a hub structure if links are to be connected sequentially, as long as forming a link requires the mutual agreement between the two involved parties.[9]

The theme of this paper is closely related to the idea of fault tolerance in computer science and electrical engineering.[10] Fault tolerance is the property that enables a system to continue operating properly in the event of the failure of some of its components. It is particularly sought-after in high-availability or life-critical systems such as the communication network, the parallel computer network and the environment network. For example, Transmission Control Protocol is designed to allow two-way communication in a packet-switched network, even in the presence of communication links which are imperfect or overloaded. As another example, for supercomputing systems, it is imperative for fault tolerance of a system to ensure that the large-scale environment is usable. To ensure such fault-tolerance, three major methods are available: (i) replicate some or the entire network, (ii) add extra stages, and (iii) add additional links. Overall, in view of fault tolerance, our concept of $\mathcal{C}^q$-stability is to require a network to tolerate q faults, i.e., it must be q-fault tolerant.[11] We achieve this by adding additional links. Therefore, our concept is equivalent to fail-stop faults in the literature.[12] By being related to fault tolerance, our paper also has some con-

sense that some nodes act as very connected hubs using a power-law distribution.

[7]The power law says that the distribution of nodes with a certain number of links follows a power function.

[8]This is also the case in Jackson and Wolinsky (1996).

[9]See Watts (2001) for a different dynamic setting. She also asserts that a star emerges with zero probability if the size of the population goes to infinity.

[10]Fault-tolerance literature is diversified on many related fields. See, for example, Adams *et al.* (1987) and Saia *et al.* (2002) on interconnection networks, Pourebrahimi *et al.* (2005) on peer-to-peer networks, and Treaster (2005) on parallel systems. For an excellent survey on fault-tolerance literature, see Avizienis *et al.* (2004).

[11]See Najjar and Gaudiot (1990).

[12]See Treaster (2005).

nection to distributed communication network literature.[13] The distributed communication network literature was spawned in 1960's when U.S. and the former Soviet Union pondered on post-nuclear attack scenarios and U.S. authorities considered ways to communicate in the aftermath of a nuclear attack. External shocks in our paper can be interpreted as nuclear attacks which can paralyze the communication network in this context.

Also, there is related literature of k-terminal reliance problem. In the classical reliance literature, the network is supposed to work when all terminal nodes – a subset of nodes – can be connected via operational links. This is a random event insofar as nodes and links may fail with some probability.[14] The k-terminal reliance problem is particularly important in areas in which the high reliability system is required. Examples include the aircraft system, the nuclear reactor control system, the defence command and control system in the military, the banking and credit certification system and telecommunications.[15] Since the k-terminal reliance problem is a probabilistic model of network reliance, the exact evaluation of the problem is extremely hard, and, in fact, is an $\mathcal{NP}$-hard problem.[16] In contrast, our model of q-connectivity is deterministic, so it is analytically tractable. Also, our problem is different from the k-reliance problem in the sense that it treats all nodes equally, not fixing a set of terminals which are basis for measuring connectivity in the k-terminal reliance problem.

The paper is organized as follows. The model is provided in the next section. In Section 3, we characterize the stable networks and the efficient networks when there is no possibility of node faults. In Section 4, we consider the possibility that some nodes are destroyed. We introduce the definitions of q-connectivity, $\mathcal{C}^q$-stability and $\mathcal{C}^q$-efficiency. Also, we describe the $\mathcal{C}^1$-stable networks and $\mathcal{C}^1$-efficient networks, and generally discuss the relation between $\mathcal{C}^q$-stability and $\mathcal{C}^q$-efficiency. Concluding remarks follow in Section 5. Formal proofs of our results are presented in the appendix.

2 Model

Let $N = \{1, 2, \cdots, n\}$ be the set of agents with $n \geq 3$. Interactions among agents are formally represented by graphs, where a node is identified with an agent, and a (direct) link

[13]See, for example, Baran (1964) for an initial work and Beaubrun and Pierre (1997) for later work.
[14]A good introductory survey can be found in Colbourn (1987) and Rubino (1998).
[15]See Rai *et al.* (1995) and Ball *et al.* (1992).
[16]See Ball (1979, 1980).

between two nodes implies that two agents can communicate directly with each other. We will denote the set of all graphs by $\mathcal{G}$.

Let g^N represent the complete graph, where there is a link between any pair of agents. If agents i and j are linked to each other in graph g, we say that they are adjacent and denote the relation by $ij \in g$. The number of links with agent i is called the degree of i and denoted by $d(i)$. A path in g between i and j is a set of distinct nodes $\{i, i_1, i_2, \cdots, i_k, j\} \subset N$ such that $ii_1, i_1i_2, \cdots, i_{k-1}i_k, i_kj \in g$. A graph g is said to be connected if there is a path between any agents i and j in N. We denote by $g + ij$ the graph obtained from adding the link ij to g and by $g - ij$ the graph obtained from deleting the link ij from g. We say that g' is a subgraph of g if $g' \subset g$. We define a subgraph of g restricted to $N'(\subset N)$ by $g|_{N'} \equiv \{ij \in g \mid i, j \in N'\}$.

Each agent pays a cost $c > 0$ of connecting each of his direct link. If agent i and j are directly linked in g, they can exchange information in one unit of time length. If agent i is indirectly linked to j in g, i.e., there is a path $\{i, i_1, \cdots, i_k, j\}$ between i and j for $k \geq 1$, there will be a delay in receiving information from each other. The delay cost can be represented by the distance between them in g which is defined by the number of links in the shortest path (geodesic) and denoted by $t(ij; g)$. If there is no path between i and j in g for some $i, j \in N$, we let $t(ij; g) = \infty$. We call $\max_j t(ij; g)$ the eccentricity of i and $\arg\max_j t(ij; g)$ the eccentric agent from i.

The total cost that each agent i incurs in graph g is the sum of the delay cost and the connecting cost, which is given by

$$w_i(g) = T_i(g) + C_i(g),$$

where $T_i(g) \equiv \sum_{j \neq i} t(ij; g)$ and $C_i(g) \equiv \sum_{j; ij \in g} c$. We will call $T_i(g)$ and $C_i(g)$ agent i's delay status and connecting status in g respectively. The total cost associated with g is $W(g) = T(g) + C(g)$, where $W(g) \equiv \sum_{i \in N} w_i(g)$, $T(g) \equiv \sum_{i \in N} T_i(g)$ and $C(g) \equiv \sum_{i \in N} C_i(g)$. We call $T(g)$ and $C(g)$ the delay status and the connecting status of the network respectively. We will use the notation of ΔT_i, ΔT, ΔW_i and ΔW for changes in various costs agent i (or the network) has to bear when the network is transformed from g to g'. For example, $\Delta T_i \equiv T_i(g') - T_i(g)$.

We will consider a situation in which each agent (node) has information that can be essential to both himself and all other agents. This requires the graph to be connected for agents to survive. We will denote the set of all connected graphs by $\mathcal{C}$. Note that $w_i(g), W(g) < \infty$ for all $i \in N$ if $g \in \mathcal{C}$, while $W(g) = \infty$ if $g \notin \mathcal{C}$.

3 Network Formation in Case of No Faults

We introduce two definitions which are adaptations from Jackson and Wolinsky (1996).

Definition 1 *A graph $g \in \mathcal{C}$ is (pairwise) stable if (i) $w_i(g) \leq w_i(g - ij)$ for all $ij \in g$ and (ii) if $w_i(g) > w_i(g + ij)$, then $w_j(g) < w_j(g + ij)$ for all $ij \notin g$.*

In words, a graph is stable if no agent has an incentive to sever a direct link unilaterally, and no pair of agents has mutual incentive to form a link between them. This definition of stability does not allow a pair of agents to sever and simultaneously form a new link à la Jackson and Wolinsky. Note that unlike Jackson and Wolinsky the stability is defined only for a connected graph. An empty graph $g = \emptyset$, for example, does not violate the conditions (i) and (ii) in Definition 1 when $n \geq 3$ since $T_i(g) = T_j(g) = T_i(g + ij) = T_j(g + ij) = \infty$, but it will not be called stable. Our interest lies only in connected graphs.

Definition 2 *Let $g_1, g_2 \in \mathcal{G}$. Then, g_1 is more efficient than g_2 and denoted by $g_1 \succ g_2$ if $W(g_1) < W(g_2)$. Also, $g \in \mathcal{C}$ is called efficient if $W(g) \leq W(g')$ for all $g' \in \mathcal{G}$.*

In words, a graph is efficient if it minimizes the sum of each agent's cost. We may compare the relative efficiency between any two graphs in $\mathcal{G}$, but obviously the efficient graph should be in $\mathcal{C}$.

The following two propositions can be easily established by invoking the proofs in the Connections Model of Jackson and Wolinsky (1996).

Proposition 1 *(i) For $c < 1$, the complete graph is the unique stable network.*

(ii) For $c > 1$, the star is stable. However, this is not unique. In particular, the circle is also stable if $\frac{n^2-4n+8}{8} < c < \frac{n(n-2)}{4}$ for $n = 4k$ or $\frac{(n-2)^2}{8} < c < \frac{n(n-2)}{4}$ for $n = 4k - 2$, or $\frac{(n-1)(n-3)}{8} < c < \frac{(n-1)^2}{4}$ for $n = 2k + 1$, and the line is also stable if $c > \frac{n(n-2)}{4}$ for $n = 2k$ or if $c > \frac{(n-1)^2}{4}$ for $n = 2k + 1$, where k is some integer.

Proof. See the appendix.

The resulting network can be either symmetric (as the complete graph or a circle) or asymmetric (as a star or a line). If the network is asymmetric, the status of agents will depend on their location in the network. For example, if the network is a star, the status of the centered agent is $(n-1)c+(n-1)$, while the status of the peripheral agent is $c+1+2(n-2)$. Therefore, as long as $c > 1$, no agent is willing to be in the center of a star. In this paper, we will sidestep this coordination problem as in usual literature.

Proposition 2 *(i) For $c < 1$, the complete graph is the unique efficient network. (ii) For $c > 1$, the star is the unique efficient network.*

Proof. See the appendix.

Jackson and Wolinsky (1996) note the tension between stability and efficiency: (i) Pairwise stability does not imply efficiency and vice versa. (ii) A stable network may be either overconnected or underconnected. This tension comes mainly from two sources. First, a selfish agent does not take into account the positive externality of forming a direct link. This is a potential source of underconnectedness. Second, it is costly for an agent to play the role of a hub in a network even if it contributes to improving efficiency. If no one can expect positive externality of a hub's many links, each agent must connect its own direct links. This can be a source of overconnectedness.

In our model, (i) the efficient network is always stable, while the converse is not the case, and (ii) a stable network is never underconnected, due to the connectivity restriction, In particular, if no one is willing to be the center of a star, a circle – which has more links than a star – can emerge as a stable network even though it is not efficient. In Jackson and Wolinsky, an efficient network may not be stable. Intuitively, this may occur when a pair of players do not want to connect a link between them even if it is more efficient to connect it, because the social benefit from a direct link is usually larger than the private benefit. In other words, the instability of an efficient network is closely related with the possibility of underconnection. The nature of overconnection in our model is one major driving force of the stability of an efficient network.

4 Network Formation in Case of Faults

In this section, we consider the possibility that one or more nodes can be destroyed for some reason. We assume that q nodes are destroyed after the network formation is completed, where q is known to all agents. Thus, agents must form networks by taking the possible future faults into account. Naturally, we are interested in the stability and the efficiency of a network which remains connected after q nodes are destroyed, that is, the robustness of the stability and the efficiency to the destruction of q nodes. We will denote by $\mathcal{C}^q$ the set of graphs connected even after the deletion of any $q(< n)$ nodes and their direct links. Some definitions of key concepts and their properties follow.

Definition 3 *A graph g is q-connected if $g \in \mathcal{C}^q$.*

In words, a graph is q-connected if it remains connected even after q nodes are deleted from it.[17] A graph will be called q-disconnected if $g \notin \mathcal{C}^q$.

Lemma 1 *(i) A graph obtained from a q-connected graph by adding a link is also q-connected. (ii) The q-connectivity of a network implies the q'-connectivity for all nonnegative integer $q' < q$.*

Proof. (i) is trivial and (ii) follows directly from (i).

The property (i) of q-connectivity suggests that there is a q-connected graph which would not be q-connected if deleting any link from it. This will be called *minimally q-connected graph*. Formally, we have

Definition 4 *A q-connected graph g is minimally q-connected if $g - ij \notin \mathcal{C}^q$ for any $ij \in g$.*

Clearly, the minimally q-connected graph may not be unique for a fixed number of population N. The following lemma provides a necessary condition for q-connectivity.

Lemma 2 *In a q-connected graph g, the degree of each node must be at least $q + 1$.*

[17] This definition is slightly different from the standard definition in graph theory. In graph theory, a graph is k-connected if it remains connected after deleting any set of fewer than k nodes. So, our definition corresponds to a $(k+1)$-connected graph in graph theory. However, we will stick to our definition because it is more intuitive at least in our setting and agrees better with the concepts of $\mathcal{C}^q$-stability and $\mathcal{C}^q$-efficiency that will be shortly defined.

Proof. See Whitney (1932).

The stability and the efficiency will be redefined among networks in $\mathcal{C}^q$.

Definition 5 *A graph $g \in \mathcal{C}^q$ is $\mathcal{C}^q$-stable if (i) $w_i(g) \leq w_i(g - ij)$ for all $ij \in g$ such that $g - ij \in \mathcal{C}^q$ and (ii) if $w_i(g) > w_i(g + ij)$, then $w_j(g) < w_j(g + ij)$ for all $ij \notin g$.*

Definition 6 *A graph $g \in \mathcal{C}^q$ is $\mathcal{C}^q$-efficient if $W(g) \leq W(g')$ for all $g' \in \mathcal{C}^q$.*

Notice that the definition of $\mathcal{C}^q$-stability requires $g - ij$ to be in $\mathcal{C}^q$. Alternatively, one could define the $\mathcal{C}^q$-stability simply by one satisfying both q-connectivity and pairwise stability. This allows a unilateral deviation possibly violating q-connectivity. We call this $\mathcal{C}^q$-strong stability. Since this concept requires a proposed network to be robust against more deviations, the set of $\mathcal{C}^q$-strongly stable networks will be usually smaller than the set of $\mathcal{C}^q$-stable networks. However, the threat to make a deviation leading to a q-disconnected graph would not be credible. Our concept of $\mathcal{C}^q$ stability requires a network to be robust against all possible *credible* deviations.[18]

The issue of connectivity has been largely unexplored in the literature of social networks, while it has been seriously considered in computer science and electric engineering as appearing in the literature of the fault tolerance issue and the k-terminal reliability problem. The concepts of $\mathcal{C}^q$-stability and $\mathcal{C}^q$-efficiency are natural generalizations merging the concept of connectivity with pairwise stability and efficiency that have been developed in the context of social networks.

Definition 7 *Two graphs g_1 and g_2 are called isomorphic and denoted by $g_1 \approx g_2$ if there is a one-to-one correspondence between the nodes of g_1 and those of g_2 so as to preserve adjacency of nodes. That is, there must be a one-to-one onto function σ mapping from the set of nodes in g_1 to the set of nodes in g_2 such that $\sigma(i)\sigma(j) \in g_2$ iff $ij \in g_1$.*

Definition 8 *Two graphs g_1 and g_2 are called iso-cost graphs and denoted by $g_1 \sim g_2$ if the cardinality of $\{(i,j) \mid t(ij; g_1) = \tau\}$ is the same as that of $\{(i,j) \mid t(ij; g_2) = \tau\}$ for all τ.*

[18] Restricting the possibility of deviations only to sustainable deviations is very common in the game theory literature. For example, the Coalition-Proof Nash equilibrium allows only for the self-enforceable joint deviations unlike the strong Nash equilibrium.

Clearly, isomorphic graphs are iso-cost graphs but the converse is not the case. If a $\mathcal{C}^q$-efficient graph has an isomorphism, the $\mathcal{C}^q$-efficient graph is not unique. Also, if a $\mathcal{C}^q$-efficient graph has an iso-cost graph, the $\mathcal{C}^q$-efficient graph is not unique. In other words, $\mathcal{C}^q$-efficiency is not unique up to isomorphism.

In the following analysis, we will pay our special attention to a "hub" of a graph. The word "hub" has been informally used to indicating a node with considerably many links relative to other nodes. We make a formal definition below.

Definition 9 *A node i is called a perfect hub of g if $ij \in g$ for all $j \in N_i$ where $N_i = N \setminus \{i\}$. A node i is called a k-imperfect hub of g if $ij \in g$ for all $j \in N_i \setminus K$ where $K = \{i_\tau \mid i_\tau \neq i, \tau = 1, \cdots, k\}$ and $ij \notin g$ for all $j \in K$.*

Indeed, if k is so large that $k \approx n$, a k-imperfect hub can be hardly considered as a hub in a practical sense. Thus, henceforth, we impose the restriction that $k < n/2$.

A hub can be found in many forms of graphs. For example, any node in g^N is a trivial perfect hub. A star has one perfect hub, but neither a circle ($n \geq 4$) nor a line ($n \geq 3$) has one. Below, we will define special forms of graphs which can have a hub (or hubs).

Definition 10 *A graph g^s and its isomorphism are called a shell if $i(i+1) \in g^s$ for $i = 1, \cdots, n-2$ and $ni \in g^s$ for $i = 1, \cdots, n-1$. The links ni for $i \neq 1, n-1$ are called teeth.*

Note that node n is a perfect hub in a shell. Figure 1 illustrates a shell.

Definition 11 *A graph $g[k]$ and its isomorphism are called a k-wheel if there is a set of nodes $K = \{i_1, \cdots, i_k\}$ such that $ij \in g[k]$ for all $i \in K$, $j \in N \setminus K$, $ij \notin g[k]$ for all $i, j \in K$ and $g[k]|_{N \setminus K}$ is a circle. We call node $i \in K$ a center of $g[k]$.*

A wheel corresponds to a 1-wheel. A center of $g[k]$ is a $(k-1)$-imperfect hub. Figure 2 illustrates a k-wheel.

Definition 12 *A graph $g^*[l]$ and its isomorphism are called an l-star if there is a set of nodes $L = \{i_1, \cdots, i_l\}$ such that $ij \in g^*[l]$ if and only if $i \in L$, $j \in N \setminus L$. Also, a graph $g^{**}[l] \equiv g^*[l] + \{ij \mid i, j \in L\}$ is called an l-perfect star. We call node $i \in L$ a center of $g^*[l]$ and $g^{**}[l]$.*

Clearly, a center of $g^{**}[l]$ is a perfect hub, while $g^{*}[l]$ is an imperfect hub, for $l \geq 2$. Figure 3 illustrates $g^{*}[l]$ and $g^{**}[l]$. A link can be called a backbone if it connects between hubs. In $g^{**}[l]$, a link ij for $i, j \in L$ is a backbone.

As we can easily see, a star is not even 1-connected. There will be two ways to strengthen connectivity. One is to connect between neighboring peripheral nodes ($g[k]$) and the other is to introduce additional centers. ($g^{*}[k+1]$) Two ways require the same number of additional links, so $g[k] \sim g^{*}[k+1]$. However, their topological property is not the same in that $g[k] \in \mathcal{C}^{k+1}$ but $g^{*}[k+1] \notin \mathcal{C}^{k+1}$.

Definition 13 *A graph $\hat{g}$ and its isomorphism are called a centered pentagon if there is a set $K = \{i_1, \cdots, i_5\}$ such that K forms a circle and for all $j \in N \setminus K$, $i_1 j, i_2 j \in \hat{g}$ for nonadjacent $i_1, i_2 \in K$.*

If $N = K$ implying that $N \backslash K = \emptyset$, the centered pentagon is the same as the circle. Note that K is the largest cycle in a centered pentagon, since i_1 and i_2 are nonadjacent. In a centered pentagon, agent i_1 and i_2 are 2-imperfect hub. In Figure 4, graph g_1 and g_2 illustrate two isomorphic centered pentagons. The graph g_3 is an iso-cost graph of g_1 and g_2, but it is not 1-connected, while g_1 and g_2 are. This means that g_3 is not isomorphic to g_1 and g_2.

For our analysis, it will be worthwhile to note the following two remarks.

Remark 1. If $c < 1$, the complete graph g^N is both the unique $\mathcal{C}^q$-stable network and the unique $\mathcal{C}^q$-efficient network for any $q < n$. (The proof follows immediately from Proposition 1, 2 and the q-connectivity of g^N.)[19]

Remark 2. For a graph g ($n \geq 3$), the followings are equivalent; (i) $g \in \mathcal{C}^1$, (ii) $g \in \mathcal{C}$ and g has no cut-node,[20] which is defined by a node whose deletion makes g disconnected, and (iii) there is a cycle through i and j for any node $i, j \in N$.[21] (The proofs can be found in a textbook on graph theory, for example, West (1996).)

[19]To elaborate on the proof, suppose that a q-connected graph $g \neq g^N$ is either $\mathcal{C}^q$-stable or $\mathcal{C}^q$-efficient for some $q < n$. Consider a graph $g' = g + ij$ for some $ij \in g^N \setminus g$. Then, clearly, $w_i(g') < w_i(g)$, $w_j(g') < w_j(g)$ and $W(g') < W(g)$ for $c < 1$, implying that g is neither $\mathcal{C}^q$-stable nor $\mathcal{C}^q$-efficient.

[20]A cut-node is illustrated in Figure 5.

[21]This is also equivalent to the statement "There are a pair of internally disjoint paths connecting i and j for any $i, j \in N$." (Whitney [1932]). Here, two paths between a pair of nodes are defined to be *internally disjoint* if neither contains a non-end node of the other.

Remark 1 allows us to concentrate only on the case that $c > 1$. Remark 2 characterizes 1-connectivity. From this, it is easy to see that neither a star nor a line is 1-connected. Remark 2, in fact, suggests that invulnerability to deletions of nodes implies multiplicity of alternative communication paths.

For the time being, let us concentrate on the case that $q = 1$.

Proposition 3 *(i) The shell cannot be $\mathcal{C}^1$-stable for any $c > 0$. (ii) The centered pentagon is $\mathcal{C}^1$-stable if $c > 1$. (iii) The circle is $\mathcal{C}^1$-stable if $c > \tilde{c}(n)$, where*

$$\tilde{c}(n) = \begin{cases} \frac{1}{8}(n^2 - 4n + 8) & \textit{for } n = 4k \\ \frac{1}{8}(n-2)^2 & \textit{for } n = 4k - 2 \\ \frac{1}{8}(n-1)(n-3) & \textit{for } n = 2k - 1. \end{cases}$$

Proof. See the appendix.

In fact, the shell is never stable, because a pair of nodes on the periphery has an incentive to delete their direct link as far as c is greater than 1 which is the increase in his delay status. On the other hand, deleting a peripheral link is not a valid deviation in $\mathcal{C}^1$-stability, but the shell cannot be $\mathcal{C}^1$-stable, either, because the hub node n has an incentive to delete one of its teeth when $c > 1$.

The centered pentagon is not stable if $c > 3$, because deleting ij for $i \in B \equiv \{i_1, i_2\}, j \in K \setminus B$ will increase the delay cost at most by 3. However, this deletion is not allowed by the definition of $\mathcal{C}^1$-stability, because the remaining graph would not be 1-connected.

Notice that a circle is more likely to be $\mathcal{C}^1$-stable than to be simply stable. This is again because the concept of $\mathcal{C}^1$-stability allows fewer possibilities of plausible deviations, although it requires an extra condition of 1-connectivity. However, the set of $\mathcal{C}^1$-strongly stable networks will be the same as the set of stability.

Now, consider $\mathcal{C}^1$-efficiency. Note that a wheel cannot be $\mathcal{C}^1$-efficient because deleting one non-spoke link would reduce the total cost at least by $2(c-1) > 0$.

Proposition 4 *(i) The shell cannot be $\mathcal{C}^1$-efficient for any $c > 0$. (ii) If a circle is $\mathcal{C}^1$-efficient for c, it is $\mathcal{C}^1$-efficient for any $c'(> c)$. (iii) A $\mathcal{C}^1$-efficient circle is $\mathcal{C}^1$-stable but not vice versa.*

Proof. See the appendix.

In a circle, each agent has exactly two direct links. This means that a circle has the minimum connection status among all 1-connected graphs. Moreover, it is unique by Lemma 3 in the appendix. But, the delay status of a circle can be large. This suggests that a graph with more direct links could be more efficient than the circle if c is small. Figure 6 illustrates $\mathcal{C}^1$-efficient graphs when $n = 7$. One can see that the efficient centered pentagon is stable, but not vice versa for $n = 7$. Also, note that a $\mathcal{C}^1$-stable network can be underconnected when $4 < c < 6$ unlike a simply stable network. Some of the illustrative results can be generalized as follows.

Theorem 1 *If $1 < c < 2$, the circle is $\mathcal{C}^1$-efficient for $n \leq 5$ and the centered pentagon is $\mathcal{C}^1$-efficient for $n > 5$.*

Proof. See the appendix.

It deserves stressing that the $\mathcal{C}^1$-efficient graph is not unique. For example, if $n = 7$ with $N = \{i \mid i = 1, \cdots, 7\}$, the graph $g = \{12, 23, 34, 45, 51\} + \{61, 63, 71, 74\}$ is also $\mathcal{C}^1$-efficient.[22]

The discussion hitherto shows that the set of stable graphs and $\mathcal{C}^q$-stable graphs are not a subset of each other and that this is the case for the set of efficient graphs and $\mathcal{C}^q$-efficient graphs. The reason is that, on one hand, $\mathcal{C}^q$-stability (or $\mathcal{C}^q$-efficiency) is more stringent due to the extra restriction of 1-connectivity, but, on the other hand, it is less stringent, because the extra requirement is also applied to possible deviations to be allowed.

Now, we will consider the possibility of multiple faults. We can see that none of the shell, the centered pentagon and the circle can be stable in the face of more than one fault, because none is q-connected ($q \geq 2$). This suggests that both $\mathcal{C}^q$-stability and $\mathcal{C}^q$-efficiency require more links added to the graphs. In fact, a graph with more than q perfect hubs is q-connected. However, we have the following proposition.

Proposition 5 *A graph with more than one perfect hub can be neither $\mathcal{C}^q$-stable nor $\mathcal{C}^q$-efficient except the complete graph for any noninteger $q < n - 2$.*

Proof. See the appendix.

[22] We are grateful to an anonymous referee for pointing out this possibility.

Definition 14 *A link $ij \in g$ is called inessential if $T_k(g') = T_k(g)$ for all $k \neq i, j$ where $g' = g - ij$ and the graph g is called multi-path if $ij \in g$ is inessential for all $ij \in g$.*

This definition says that if a link is inessential, deletion of the link does not affect the status of all other nodes than its two ends. If, for each pair indirectly connected, there is an alternative path of the same length, each link is inessential.

Proposition 6 *(i) A $\mathcal{C}^q$-efficient network g must be minimally q-connected if $c > 2$ and g is multi-path. (ii) A $\mathcal{C}^q$-efficient network g is $\mathcal{C}^q$-stable if g is minimally q-connected. (iii) A $\mathcal{C}^q$-efficient network g is $\mathcal{C}^q$-stable if g is multi-path.*

Proof. See the appendix.

Proposition 6(i) gives a necessary condition for $\mathcal{C}^q$-efficiency. In addition, Proposition 6(ii) - (iii) provide the general relation between $\mathcal{C}^q$-stability and $\mathcal{C}^q$-efficiency. They give sufficient conditions for a $\mathcal{C}^q$-efficient network to be $\mathcal{C}^q$-stable. Although the multi-path condition is rather strong, it is satisfied in many graphs including shell, k-wheel and l-star.

Before closing this section, we will briefly discuss the possibility of a backbone link. The simple graphs described in this paper like k-wheel or l-perfect star have indirect links of length at most 2. Thus, all hubs are connected in length 2 via peripheral nodes. Thus, a backbone link is inessential and inefficient as long as $c > 1$. Of course, a backbone link could improve efficiency if the length between a hub and a peripheral node exceeds 1.

5 Conclusion

In this paper, we discussed the $\mathcal{C}^q$-stability and the $\mathcal{C}^q$-efficiency of networks when there is a possibility that q nodes will be destroyed.

Generally, identifying the $\mathcal{C}^q$-efficient network is very difficult, and beyond the scope of this paper. Since it requires searching for the global optimum, it involves complex numerical computations. However, it will be a challenging future research issue to derive the general principle determining the shape of $\mathcal{C}^q$-efficient network.

Appendix

Proof of Proposition 1: (i) Let us start with the complete graph. Suppose i severs a link with j for any $i, j \in N$. In this graph g', $t(ij; g') = t(ij; g^N) + 1$, and thus $w_i(g') = w_i(g^N) + 1 - c$. Therefore, we have $w_i(g') > w_i(g^N)$ if $c < 1$. This implies that agent i has no incentive to delete a link to j for any $i, j \in N$. Thus, g^N is stable. Now, consider a network g which is not complete. Since $g \neq g^N$, there must be a pair (i, j) such that $ij \notin g$. Consider a graph $g'' = g + ij$. Then, it is easy to see that i and j have a mutual incentive to form a link between them. Since $t(ij; g'') \leq t(ij; g) - 1$ and $t(il; g'') \leq t(il; g)$ for all $l \in N \setminus \{i, j\}$, we have $w_i(g'') \leq w_i(g) - 1 + c < w_i(g)$. Hence, i has an incentive to form a direct link with j and so does j. This implies that no graph other than g^N is stable if $c < 1$. This establishes the uniqueness.

(ii) First, we will prove that the star g is stable if $c > 1$. Consider $ij \in g$. Neither i nor j has an incentive to sever the link ij, because $g - ij \notin \mathcal{C}$. Now, consider $g' = g + ij$ such that $ij \notin g$. Then, $t(ij; g') = t(ij; g) - 1$. Thus, $w_l(g') = w_l(g) - 1 + c > w_l(g)$ for $l = i, j$, implying that i and j have no incentive to link themselves directly. Thus, the star is stable.

Second, we prove the stability of a circle. Pick an arbitrary i on a circle g. If he severs a link ij such that $ij \in g$, then $g' = g - ij$ is a line. Then, we have

$$\Delta T_i = \sum_{l=1}^{\frac{n-2}{2}} 2l = \frac{1}{4} n(n-2) \tag{1}$$

for an even $n \geq 4$. So, for stability of g, it must be that $c < \frac{n(n-2)}{4}$. Also, we need to check the incentive to form a link (or links). If agent i has no incentive to form a link to his eccentric agent j, neither does he to any other agent and, moreover, no more than one pair will be linked, either. So, it is sufficient to check the incentive to add a new link ij. For agent i (or agent j), we can calculate the gain in the delay status as

$$-\Delta T_i = \begin{cases} \left(\frac{n}{2} - 1\right) + 2\sum_{l=1}^{\frac{n}{4}-1} (2l - 1) = \frac{1}{8}(n^2 - 4n + 8), & \text{if } n = 4k \\ \left(\frac{n}{2} - 1\right) + 2\sum_{l=1}^{\frac{n}{4}-\frac{3}{2}} 2l = \frac{1}{8}(n-2)^2, & \text{if } n = 4k - 2. \end{cases} \tag{2}$$

Hence, the pair will not form a link if $c > \frac{n^2-4n+8}{8}$ for $n = 4k$ and if $c > \frac{(n-2)^2}{8}$ for $n = 4k - 2$. Hence the circle is stable for an even n under the specified condition.

Now, consider the case that $n = 2k + 1$. If a link is formed between i and his eccentric agent for an odd n, his gain in the delay status can be computed as

$$-\Delta T_i = \sum_{l=1}^{k-1} l = \frac{k(k-1)}{2} = \frac{1}{8}(n-1)(n-3). \tag{3}$$

Hence the pair will not form a link if $c > \frac{1}{8}(n-1)(n-3)$ for an odd n. The deletion will lead to

$$\Delta T_i = \sum_{l=1}^{k}(2l-1) = k^2 = \frac{1}{4}(n-1)^2. \tag{4}$$

Therefore the stability of a circle for $n = 2k + 1$ is also proved.

Finally consider a line. If a pair of agents in both ends of the link are not willing to form a link between them, no other pair will be, either. Consider a link between agents in both ends. From formulas (1) and (4), we can easily see that they do not form a link if $c > \frac{n(n-2)}{4}$ for an even n and if $c > \frac{(n-1)^2}{4}$ for an odd n. Clearly, no agent is willing to sever a link.

Proof of Proposition 2: (i) Suppose $ij \notin g$. Now, consider $g' = g + ij$. Then, $T_k(g') \leq T_k(g) - 1$, i.e., $\Delta T_k \leq -1$, thus $\Delta W_k \leq c - 1 < 0$, if $c < 1$, for $k = i, j$. Also, we have $\Delta W_k \leq 0$ for all $k \in N \setminus \{i, j\}$. Thus, $\Delta W < 0$. Since this argument holds for all $ij \notin g$, the result is immediate.

(ii) A pair of agents in g is either directly linked or indirectly linked. The cost associated with the direct links between any pair of agents is $2(1 + c)$. On the other hand, the cost associated with the indirect links of length $k(\geq 2)$ is $2k$. Since $c > 1$, the cost from a direct link unambiguously exceeds the cost from an indirect link of length 2. This implies that the total cost associated with g is minimized when g is connected by the smallest number of direct links and the length of all indirect links is 2. Also, the connectivity of g requires at least $n - 1$ direct links. Thus, the star is the only network satisfying the three conditions, implying that the star is the unique efficient network.

Proof of Proposition 3: (i) Consider $g = g^s - ni$ for some $i \in N \setminus \{n\}$. Since $\Delta T_n = 1$, it is clear that $w_n(g) < w_n(g^s)$ if $c > 1$, implying that node n has an incentive to sever the link ni. Since g^N is the unique $\mathcal{C}^1$-stable when $c < 1$, the proof is completed.

(ii) Since the maximum length between two agents is 2, no pair of agents has the incentive to add a link as long as $c > 1$. Also, since $\hat{g}$ is minimum 1-connected by Lemma 2, the proof is completed.

(iii) This follows directly from Proposition 1, since the circle is 1-connected but a line is not 1-connected.

Proof of Proposition 4: (i) For given n, consider two graphs $g_1 = g^s$ and $g_2 = g^{**}[2]$, both of which are 1-connected. Then, $g_1 \sim g_2$ because they have the same number of direct links, $(n-1)+(n-2) = 2n-3$, and indirect links of length 2, $\binom{n}{2} - (2n-3)$. However, g_2 cannot be $\mathcal{C}^1$-efficient when $c > 1$ because $g_2 - ij \succ g_2$ where ij is a backbone. So, g_1 cannot be $\mathcal{C}^1$-efficient, either.

Lemma 3 *For $n \geq 3$, the circle is the unique network each of whose node has exactly two direct links.*

Proof of Lemma 3: Consider an arbitrary node of the graph g, say 1. Label two nodes adjacent to node 1 as node 2 and node 3. If $n = 3$, the proof is done because it must be that $23 \in g$. Suppose $n \geq 4$. Then, $23 \notin g$. Otherwise, $g \notin \mathcal{C}$. Label the node directly linked to node 2 by node 4. If $n = 4$, g is a circle because it must be that $34 \in g$. If $n \geq 5$, $34 \notin g$. Label the node directly linked to node 3 as node 5. If $n = 5$, $45 \in g$, and if $n > 5$, $45 \notin g$. This argument can be repeated. Generally, if $n = k$, it must be that $(k-1)k \in g$. Thus, g is a circle.

(ii) Let g be a circle. We have $C(g) = 2nc$ and $C(g') = 2(n+k)c$ for some $k \geq 1$ by Lemma 3 for $g'(\neq g) \in \mathcal{C}^1$. $\mathcal{C}^1$-efficiency of a circle for c requires that

$$2nc + T(g) \leq 2(n+k)c + T(g'), \forall g' \in \mathcal{C}^1,$$

implying that $c \geq \Delta T(g')/2k$ for all $g' \neq g$, where $\Delta T(g') = T(g') - T(g)$. Therefore, $c' \geq c \geq \Delta T(g')/2k$ for all $g'(\neq g) \in \mathcal{C}^1$.

(iii) Recall from Proposition 3(iii) that the condition for the $\mathcal{C}^1$-stability of a circle is $c > \tilde{c}(n)$. To obtain the condition for the $\mathcal{C}^1$-efficiency of a circle, we only need to compute ΔW by adding a link between any node i and its eccentric node j. It is easy to see that adding ij increases the connecting status of the graph by $2c$ and decreases the delay status by at least $2\tilde{c}(n)$, since the delay cost of some node $k \neq i, j$ may also be decreased by the link. Thus, we have $\Delta W = 2c - (2\tilde{c}(n) + \alpha)$ for some $\alpha \geq 0$ with equality if $n = 4, 5, 6$, implying that a circle is $\mathcal{C}^1$-efficient if $\Delta W > 0$, or equivalently, $c > \tilde{c}(n) + \alpha/2 \equiv \underline{c}(n)$. Since $\underline{c}(n) \geq \tilde{c}(n)$, a $\mathcal{C}^1$-efficient circle is $\mathcal{C}^1$-stable, but not vice versa.

Proof of Theorem 1: Denoting a $\mathcal{C}^1$-efficient graph by g^*, we have

Lemma 4 $\max_{ij \in g^*} t(ij; g^*) = 2$.

Proof. If $t(ij) \geq 3$ for some $ij \in g^*$, $g^* + ij \succ g^*$ since $c < 2$. Contradiction. If $t(ij) = 1$ for all $i, j \in N$, we can delete a link ij and $g^* - ij \succ g^*$ since $c > 1$. Contradiction.

Lemma 5 *g^* has the minimum number of links in the set $G_2 \equiv \{g \in \mathcal{C}^1 \mid t(ij; g) \leq 2, \forall ij \in g\}$.*

Proof. By Lemma 4, $t(ij) = 1$ or 2 for any $ij \in g^*$. The cost associated with a direct link is $c+1$, while the cost associated with an indirect link is 2. Since $c+1 > 2$, efficiency requires the minimum number of direct links.

We will call $g \in G_2$ a graph spanning N or simply a spanning graph. Then, Lemma 5 says that the $\mathcal{C}^1$-efficient graph has the smallest number of links among all spanning graphs.

We will prove Theorem 1. It is easy to see that g^* is the circle if $n = 4$ from Lemma 5. Since the circle is identical to the centered pentagon when $n = 5$, it suffices to show that $|g^*| = |\hat{g}|$ for all $n \geq 5$.

Let $s \in N$ denote a node with the smallest degree in the $\mathcal{C}^1$-efficient graph g^*. We only need to consider the case that $d(s; g^*)$ is either 2 or 3, since $d(s; g^*) \geq 4$ implies that $|g^*| \geq \frac{4n}{2} = 2n > 2n - 5 = |\hat{g}|$, yielding a contradiction.

Case 1 **($d(s) = 2$)** Let u, v be two adjacent nodes to s in g^*, i.e., $us, vs \in g^*$. Define $N' = N \setminus \{s, u, v\}$. Then, all nodes $i \in N'$ must be linked to either u or v or both, since $t(is; g^*) \leq 2$ for all $i \in N'$. Let Γ_k for $k = 1, 2$ denote the set of nodes in N' which have k links to $\{u, v\}$ and let $\gamma_k = |\Gamma_k|$.

If $\Gamma_1 = \emptyset$, the $\mathcal{C}^1$-efficient graph can be obtained when all nodes in Γ_2 has only two links. Since $\gamma_2 = n - 3$, we have $|g^*| = 2 + 2\gamma_2 = 2n - 4 > 2n - 5 = |\hat{g}|$, which is a contradiction to the $\mathcal{C}^1$-efficiency of g^*. Thus, it must be that $\Gamma_1 \neq \emptyset$.

Let $u', v' \in \Gamma_1$ be arbitrary nodes such that $u'u \in g^*$ and $v'v \in g^*$ respectively. For the time being, assume that any $z \in \Gamma_2$ has only two links, i.e., $u'z, v'z \notin g^*$ for any $z \in \Gamma_2$ when $\Gamma_1 \neq \emptyset$. By Whitney (1932), 1-connectivity of g^* implies that there are a pair of internally disjoint paths between u' and v' for any $u', v' \in \Gamma_1$, i.e., there must be a path between u' and

v' not including u, s nor v. Since any $z \in \Gamma_2$ has no other links than uz and vz, this implies that $g' \equiv g^*|_{\Gamma_1}$ is connected. Note that g' has the minimum number of links when it is a tree. Thus, $|g'| \geq \gamma_1 - 1$ for all $\gamma_1 \geq 1$. In fact, a graph g spanning N can have the minimum number of links when nodes in Γ_1 forms a tree and all nodes in Γ_2 have only two links. Therefore, we have $|g^*| \geq 2 + \gamma_0 + \gamma_1 + 2\gamma_2 + (\gamma_1 - 1) = 1 + \gamma_0 + 2(\gamma_1 + \gamma_2) = 1 + \gamma_0 + 2(n-3) = 2n - 5 + \gamma_0$ where $\gamma_0 = 1$ if $uv \in g^*$ and $\gamma_0 = 0$ if $uv \notin g^*$. This means that $\hat{g} \leq g^*$ for any $\mathcal{C}^1$-efficient graph g^*.

Case 2 ($d(s) = 3$) Let u, v, w be three adjacent nodes to s in g^*, i.e., $us, vs, ws \in g^*$. Also, let Γ_k for $k = 1, 2, 3$ denote the set of nodes in $N \setminus \{s, u, v, w\}$ which have k links to $\{u, v, w\}$ and $\gamma_k = |\Gamma_k|$. If $\Gamma_1 = \Gamma_2 = \emptyset$, we have $|g^*| = 3 + \gamma_0 + 3(n-4) > 3 + 2(n-4) = 2n - 5$ for $n \geq 5$ where γ_0 is the number of direct links among $\{u, v, w\}$. Contradiction. Thus, we must have $\Gamma_1 \neq \emptyset$ or $\Gamma_2 \neq \emptyset$.

(i) Consider the case that $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$. By the same argument as in case 1, there must be a path P between any pair of nodes in Γ_1 such that $u, v, w, s \notin P$ by Whitney (1932). Similarly, there must be a path P' between any node in Γ_1 and any node in Γ_2 such that $u, v, w, s \notin P'$. If any node in Γ_3 has no other link than the three links to u, v and w, this implies that $g'' \equiv g^*|_{\Gamma_1 \cup \Gamma_2}$ is connected. Note that a spanning graph can have the minimum number of links when all nodes in Γ_3 have exactly three links and g'' is a tree. Then, it follows that $|g''| \geq (\gamma_1 + \gamma_2) - 1$ with the equality if g'' is a tree, since $\Gamma_1 \cup \Gamma_2 \neq \emptyset$. Therefore, we have $|g^*| \geq 3 + \gamma_0 + \gamma_1 + 2\gamma_2 + 3\gamma_3 + |g''| \geq 2 + \gamma_0 + 2(\gamma_1 + \gamma_2 + \gamma_3) + \gamma_2 + \gamma_3 \geq 2n - 5 = |\hat{g}|$, since $\gamma_2 \geq 1$.

(ii) If $\Gamma_1 \neq \emptyset$ and $\Gamma_2 = \emptyset$, all nodes in Γ_1 must have at least two more links, since $d(i) \geq 3$ for all $i \in N$. Thus, a spanning graph has the minimum number of links when nodes in Γ_1 form a circle, requiring γ_1 links. Therefore, we have $|g^*| \geq 3 + \gamma_0 + \gamma_1 + 3\gamma_3 + \gamma_1 = 3 + \gamma_0 + 2(n-4) + \gamma_3 = 2n - 5 + \gamma_0 + \gamma_3 \geq 2n - 5 = |\hat{g}|$.

(iii) Finally, if $\Gamma_1 = \emptyset$ and $\Gamma_2 \neq \emptyset$, we have $|g^*| = 3 + \gamma_0 + 2\gamma_2 + 3\gamma_3 + \alpha$ for some α. Because $d(s) = 3$, any $i \in \Gamma_2$ must have at least one more link. Hence, $\alpha \geq 1$. Thus, we have $|g^*| = 3 + \gamma_0 + 2(n-4) + \gamma_3 + \alpha > 2(n-5)$, which is contradiction. This completes the proof.

Proof of Proposition 5: If $c < 1$, the result is immediate from Proposition 1 and 2. Consider the case that $c > 1$. Suppose g is $\mathcal{C}^q$-stable with the set of hubs denoted by

$H = \{i_1, i_2, \cdots, i_k\}$ for $k \geq 2$. Deleting ij for $i, j \in H$ will increase their delay cost by one instead of saving c, while $g - ij$ is still q-connected. So, they would have the incentive to sever the link if $c > 1$, implying that g is not $\mathcal{C}^q$-stable. Now, suppose g is $\mathcal{C}^q$-efficient. Similarly, $g - ij \succ g$. So, g cannot be $\mathcal{C}^q$-efficient, either.

Lemma 6 *$t(ij; g - ij) - t(ij; g) \leq 2$ for all $ij \in g$ if g is multi-path.*

Proof. Suppose that $t(ij; g - ij) - t(ij; g) \geq 3$ for some $ij \in g$. Let $g' = g - ij$. Also, let the geodesic between i and j in g' be $K \equiv \{i, k_1, \cdots, k_r, j\} \subset N$ where $r \geq 3$. Then, $t(jk_1; g') \geq 3 \neq t(jk_1; g) = 2$. Thus, ij is not inessential, implying that g is not multi-path. This is a contradiction.

Proof of Proposition 6: (i) Suppose g is not minimally q-connected. Then, it must be that $g - ij \in \mathcal{C}^q$ for some $ij \in g$. If we let $g' = g - ij$, we have $t(ij; g') - t(ij; g) \leq 2$ by Lemma 6. If $t(ij; g') - t(ij; g) = 1$, it is clear that $g' \succ g$, since $c > 2 > 1$. If $t(ij; g') - t(ij; g) = 2$, it is also clear that $g' \succ g$, because $c > 2$. Thus, g is not $\mathcal{C}^q$-efficient. Contradiction.

(ii) Since g is the minimally q-connected graph, $g - ij \notin \mathcal{C}^q$ for all $ij \in g$. Suppose g is not $\mathcal{C}^q$-stable. Then, for some pair of (i, j), $w_i(g + ij) < w_i(g)$ and $w_j(g + ij) \leq w_j(g)$. Consider $g' = g + ij$. Then, it is clear that $g' \in \mathcal{C}^q$ and $g' \succ g$, implying that g is not $\mathcal{C}^q$-efficient. This is a contradiction.

(iii) Suppose g is not $\mathcal{C}^q$-stable. Then, there must be a pair of (i, j) such that either (A) $w_i(g + ij) < w_i(g)$ and $w_j(g + ij) \leq w_j(g)$ or (B) $w_i(g) > w_i(g - ij)$ and $w_j(g) > w_j(g - ij)$ where $g - ij \in \mathcal{C}^q$. (In both cases, $\Delta w_i = \Delta w_j$ since g is multi-path.) If (i, j) satisfies (A), clearly $g + ij \succ g$, which is a contradiction to $\mathcal{C}^q$-efficiency of g. If (i, j) satisfies (B), $g - ij \succ g$ as long as ij is inessential. Thus, the proof is completed.

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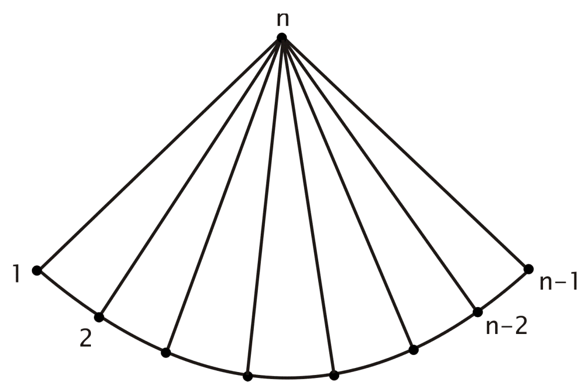


Figure 1: A shell graph

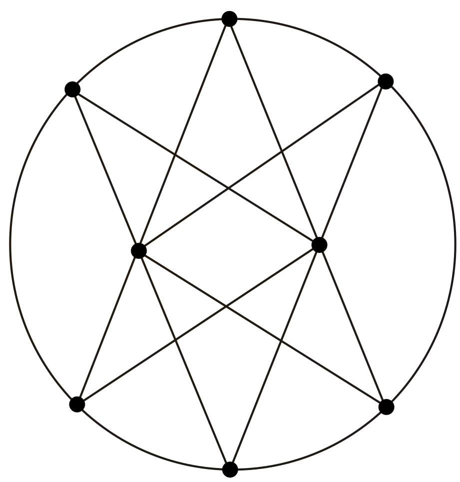


Figure 2: 2-wheel

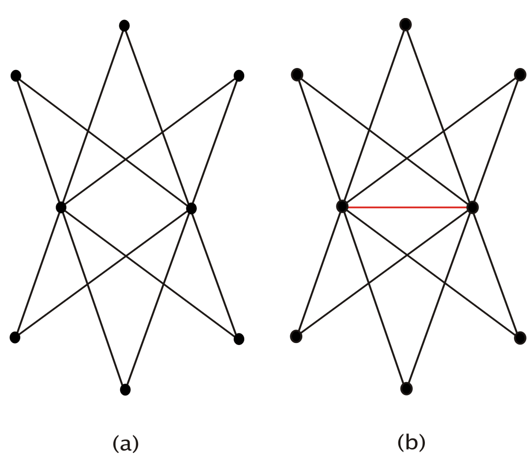


Figure 3: $g^*[l]$ (left) and $g^{**}[l]$ (right)

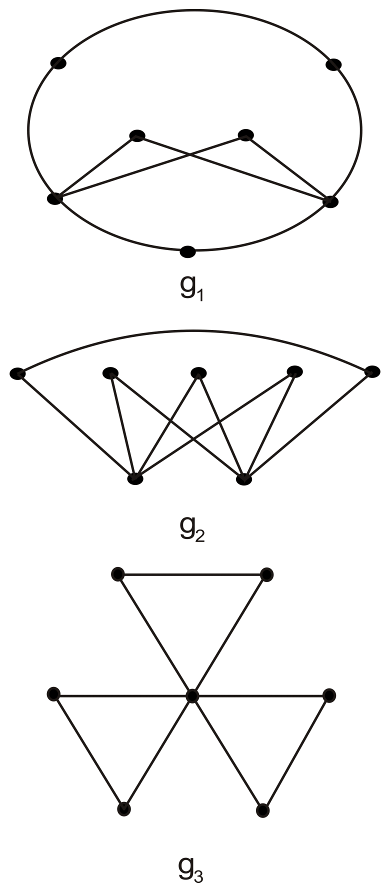


Figure 4: Two isomorphic centered pentagons and their iso-cost graph

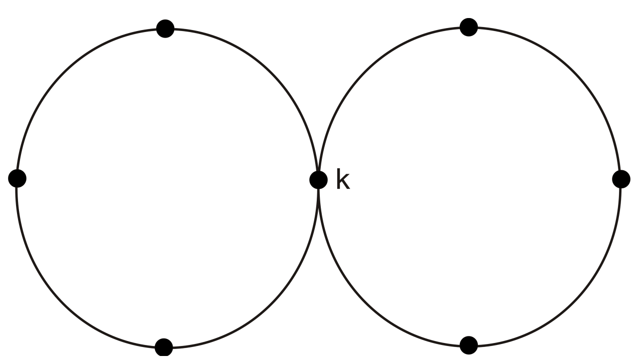
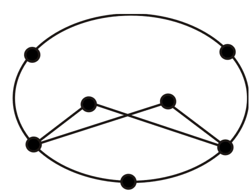
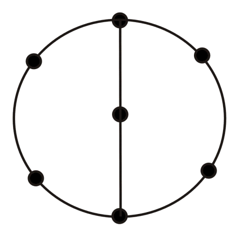


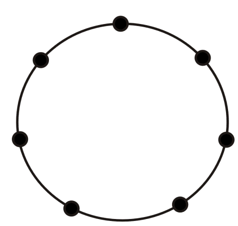
Figure 5: A graph with a cut-node (node k)



(a) 1<c<3



(b) 3<c<6



(c) c>6

Figure 6: $\mathcal{C}^1$-efficient graphs for $n = 7$